KEMENTERIAN
PENDIDIKAN
MALAYSIA

DUAL LANGUAGE PROGRAMME

ADDITIONAL MATHEMATICS

Form 4

KSSM
KURIKULUM STANDARD SEKOLAH MENENGAH
KEMENTERIAN PENDIDIKAN MALAYSIA

RUKUN NEGARA

Bahawasanya Negara Kita Malaysia
mendukung cita-cita hendak;

Mencapai perpaduan yang lebih erat dalam kalangan
seluruh masyarakatnya;

Memelihara satu cara hidup demokrasi;

Mencipta satu masyarakat yang adil di mana kemakmuran negara
akan dapat dinikmati bersama secara adil dan saksama;

Menjamin satu cara yang liberal terhadap
tradisi-tradisi kebudayaannya yang kaya dan pelbagai corak;

Membina satu masyarakat progresif yang akan menggunakan
sains dan teknologi moden;

MAKA KAMI, rakyat Malaysia,
berikrar akan menumpukan
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KEDAULATAN UNDANG-UNDANG
KESOPANAN DAN KESUSILAAN

(Sumber: Jabatan Penerangan, Kementerian Komunikasi dan Multimedia Malaysia)

KURIKULUM STANDARD SEKOLAH MENENGAH

ADDITIONAL MATHEMATICS

FORM 4

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- *Desmos*
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Introduction

This **KSSM Form 4 Additional Mathematics Textbook** is written based on *Dokumen Standard Kurikulum dan Pentaksiran* (*DSKP*) for Additional Mathematics Form 4. The purpose of Additional Mathematics *Kurikulum Standard Sekolah Menengah* (*KSSM*) is to form individuals who think mathematically, creatively and innovatively as well as having a good image.

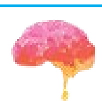
The contents of this textbook integrate six basis of the *KSSM* structure, knowledge, skills and values as well as explicitly instilling 21st Century Skills and Higher Order Thinking Skills (HOTS). This textbook weaves the diversity of teaching and learning strategies to enable students to understand the contents in-depth as well as sharpen their thinking to a higher level. Through the complete usage of this book, students will actively participate in inquiry-based learning that involves experience, investigation and exploration.

Cross-Curricular Elements (CCE) such as the correct usage of language medium, moral values, patriotism, scientific and technological literacy, creativity and innovation, entrepreneurship, information technology and financial education are wholly instilled in the formation of contents of this textbook. A STEM approach is also applied in this book in preparation for students to face challenges and be competitive at the global stage.

Special Characteristics in this Book and its Functionalities

Stimulus Pages
- Contain interesting photographs and texts that relate to daily life which stimulate students' thinking.
- Contain Content Standards in 'What will be learnt?', learning aims in 'The significance of this chapter', history or general information about the chapter in 'Did you know?' and bilingual Key Words.

	The QR code on the front cover of the book contains explanation of the book themes, the author's biography as well as updated information and facts (if any).
INQUIRY 1 In pairs / In groups / Individual	Activities that involve students individually, in pairs or in groups which encourage students to be actively involved in the learning process.
Self Practice 1.1	Prepares questions to test the students' understanding about learnt concepts.
Intensive Practice 1.1	Contains questions to determine students' mastery of a learnt topic.

MATHEMATICS APPLICATION	Gives problem solving questions as well as working steps that cover real life situations.
FLASHBACK	Shows information that is learnt by students.
Mind Challenge	Provides questions that require students to think creatively and test students' performance.
Mathematics Museum	Gives explanation about the developments in the history of mathematics and contributions of mathematicians.
BRAINSTORMING	Provides activities that require discussion amongst students.
QUICK COUNT	Explains the ways of using the scientific calculator in mathematical calculations.
Tech Whizz	Gives exposure to students regarding applications of technology in mathematical learning.
QR	Gives exposure to students using mobile devices by scanning the QR code to get additional information.
Smart TIPS	Gives mathematical tips that relate to the topic for student's use.
Alternative Method	Suggests an alternative method to certain questions.
MATHEMATICS POCKET	Gives additional information to students to master the learnt topic more.
SUMMARY OF CHAPTER	A whole coverage about the learnt chapter.
WRITE YOUR JOURNAL	Apply learnt concepts in daily lives.
Exploring MATHEMATICS	Brief activities that relate to the learnt topic.
MASTERY PRACTICE	Covers LOTS and HOTS questions to test students' understanding.
	HOTS question is to stimulate students' higher order thinking skills.
21st Century Learning	Uses 21st century learning concepts to increase students' understanding.
1.1.2	Shows the learning standards for each chapter.
PL1	Shows the performance level for each question.

Contents

Formulae

Chapter 2 Quadratic Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chapter 4 Indices, Surds and Logarithms

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$

$\log_a mn = \log_a m + \log_a n$

$\log_a \frac{m}{n} = \log_a m - \log_a n$

$\log_a m^n = n \log_a m$

$\log_a b = \frac{\log_c b}{\log_c a}$

Chapter 5 Progression

Arithmetic Progressions

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_n = \frac{n}{2}[a + l]$

Geometric Progressions

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r}, |r| < 1$

$S_n = \frac{a(r^n - 1)}{r - 1}, |r| > 1$

$S_\infty = \frac{a}{1 - r}, |r| < 1$

Chapter 7 Coordinate Geometry

Divisor of a Line Segment

$= \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}\right)$

Area of triangle

$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$

Area of quadrilaterals

$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1)$

$- (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)]$

Chapter 8 Vectors

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$

$\hat{\underset{\sim}{r}} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$

Chapter 9 Solution of Triangles

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Area of triangle

$= \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$

Heron's Formula

$= \sqrt{s(s - a)(s - b)(s - c)}, s = \frac{a + b + c}{2}$

Chapter 10 Index Number

$I = \frac{Q_1}{Q_0} \times 100$

$\bar{I} = \frac{\sum I_i w_i}{\sum w_i}$

bit.ly/2nFGX42

Download the free QR code scanner application from *Google Play*, *App Store* or other platforms to your mobile devices. Scan the *QR* Code using the application or visit the website as shown on the left to download PDF files, *GeoGebra* and complete answers. Then, save the downloaded files for offline use.

CHAPTER 1

Functions

What will be learnt?

- Functions
- Composite Functions
- Inverse Functions

List of Learning Standards

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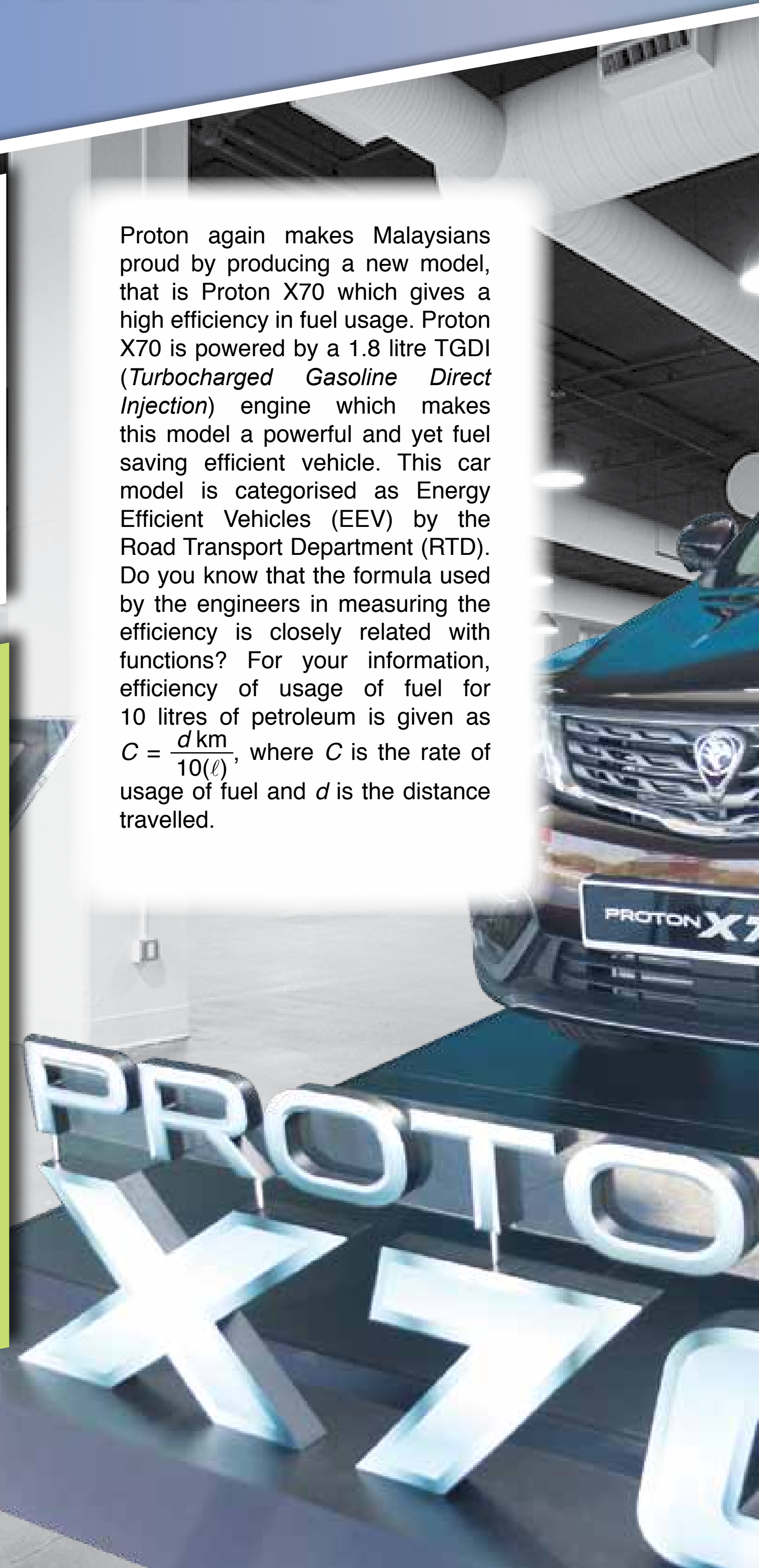
KEYWORDS

Function notation	*Tatatanda fungsi*
Undefined function	*Fungsi tidak tertakrif*
Absolute value function	*Fungsi nilai mutlak*
Vertical line test	*Ujian garis mencancang*
Arrow diagram	*Gambar rajah anak panah*
Object	*Objek*
Image	*Imej*
Domain	*Domain*
Codomain	*Kodomain*
Range	*Julat*
Discrete function	*Fungsi diskret*
Continuous function	*Fungsi selanjar*
Composite function	*Fungsi gubahan*
Inverse function	*Fungsi songsang*
Horizontal line test	*Ujian garis mengufuk*



Proton again makes Malaysians proud by producing a new model, that is Proton X70 which gives a high efficiency in fuel usage. Proton X70 is powered by a 1.8 litre TGDI (*Turbocharged Gasoline Direct Injection*) engine which makes this model a powerful and yet fuel saving efficient vehicle. This car model is categorised as Energy Efficient Vehicles (EEV) by the Road Transport Department (RTD). Do you know that the formula used by the engineers in measuring the efficiency is closely related with functions? For your information, efficiency of usage of fuel for 10 litres of petroleum is given as $C = \frac{d\text{ km}}{10(\ell)}$, where C is the rate of usage of fuel and d is the distance travelled.

Did you Know?

The subject of function was first introduced by the French mathematician, Rene Descartes, in the year 1637. According to him, a function is any variable *x* where its power is a positive integer.

However, Leonhard Euler (1707-1783), a mathematician from Switzerland stated that a function is any equation or formula involving variables and constants. His idea regarding functions is similar to what is being studied these days.

For further information:

bit.ly/2B2y33v

SIGNIFICANCE OF THIS CHAPTER

A function gives rise to simple and accurate mathematical model in representing a situation as well as in solving problems faced in our surroundings. For example:

- The height of an individual, *h*, is a function related to his/her thigh bones, *f*. By substituting the values of *f* into the function *h*, forensic experts are able to estimate the height of a corpse based on his/her thigh bones.
- Bank officers use the concept of functions in calculating the interest incurred in a loan and hence in the instalments of purchasing a house, a car, a personal or business loan of their clients.

Scan this QR code to watch a video on Proton X70.

bit.ly/2Rnu0Zh

1.1 Functions

There are many quantities which depend on one or more variables in our daily lives. Study and understand the following situations:

You are working as a temporary cashier and is paid RM80 daily. The total payment received is determined by the number of days you worked.

You are buying durians from a stall. If the cost of one kilogram of durian is RM8, the total amount you need to pay depends on the weight of durians you bought.

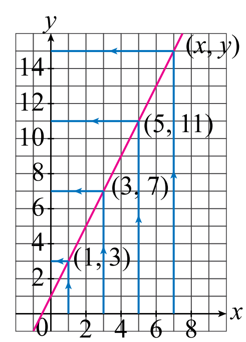
In mathematics, such situations are examples of functions. From the above examples of situations, state the meaning of a function.

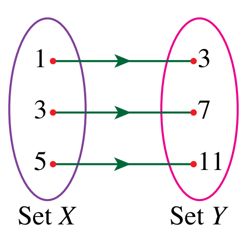
Explaining functions by graphical representation and notation

Look at the graph of $y = 2x + 1$ on the right. The relation between the value of 1 on the x-axis and the value of 3 on the y-axis can be written as $1 \rightarrow 3$. This indicates that 1 is the first element and 3 is the last element.

In this case, we can say that 1 is mapped to 3. Similar to $3 \rightarrow 7$, $5 \rightarrow 11$ and so on. Every point (x, y) on the line is corresponding to the mapping of $x \rightarrow y$ where x on the x-axis is mapped to y on the y-axis.

The relation from part of the mapping $x \rightarrow y$ can be represented by an arrow diagram as shown below.





Each element x in set X is mapped to one and only one element y in set Y.

Thus, this type of relation is known as **function** or **mapping**.

In general:

Function relating set X to set Y is a special relation where each element $x \in X$ is mapped to one and only one element $y \in Y$

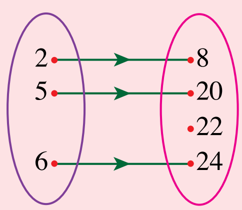
If f is a function from set $X = \{1, 3, 5\}$ to set $Y = \{3, 7, 11\}$ and is defined by $f : 1 \rightarrow 3$, $f : 3 \rightarrow 7$ and $f : 5 \rightarrow 11$, element 1 is known as the object and element 3 is its image. Similarly, 7 and 11 are the images of 3 and 5 respectively. Any element x in set X that is mapped to one element y in set Y by $y = 2x + 1$ is written in function notation as below:

$f : x \rightarrow y$ or $f(x) = y$
$f : x \rightarrow 2x + 1$ or $f(x) = 2x + 1$
where x is the object and $2x + 1$ is the image

Example 1

Are the following relations a function? Explain.

(a)



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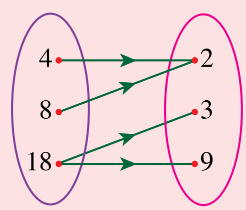

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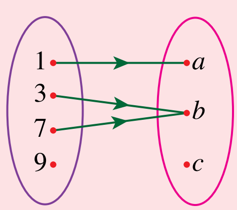

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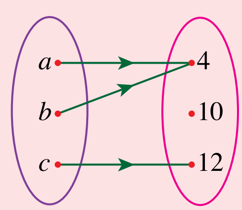
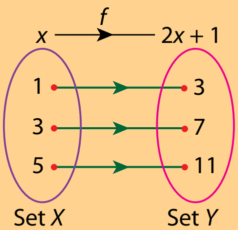


Solution

(a) This relation is a function because each object has only one image even though element 22 has no object.
(b) This relation is not a function because it does not satisfy the condition of being a function, that is each object has only one image. Note that 18 has two images, that are $18 \rightarrow 3$ and $18 \rightarrow 9$.
(c) This relation is not a function because it does not satisfy the condition of being a function, that is each object has only one image. Note that 9 does not have any image.
(d) This relation is a function because each object has only one image even though element 10 has no object.

MATHEMATICS POCKET

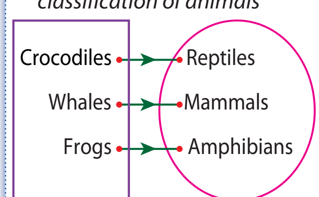
- $f : x \rightarrow 2x + 1$ is read as "function f maps x to $2x + 1$".
- $f(x) = 2x + 1$ is read as "$2x + 1$ is the image of x under the function f" or "the function f of x is equal to $2x + 1$".
- 

is read as "$2x + 1$ for 1 is 3" and so on.

FLASHBACK

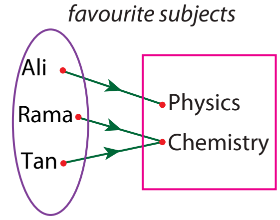
Function is a relation of one-to-one or a relation of many-to-one.

classification of animals



Relation of one-to-one

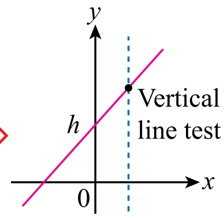
favourite subjects

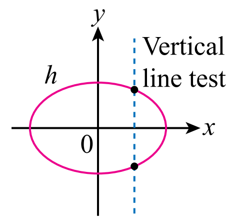


Relation of many-to-one

How can we determine whether a graph of a relation is a function? When a graph is given, we use the **vertical line test** to determine whether the given graph is a function or otherwise. If the vertical line cuts only at one point on the graph, then the relation is a function. On the other hand, if the vertical line does not cut the graph at any point or cuts more than one point, then the graph is not a function.

The graph of h is a function.



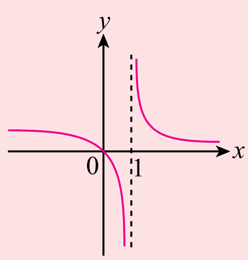


The graph of h is not a function.

Example 2

Which of the following graph represents a function?

(a)



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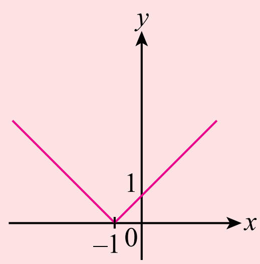
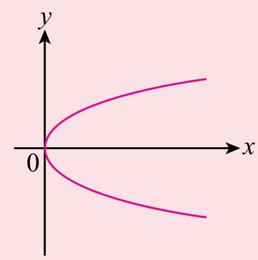
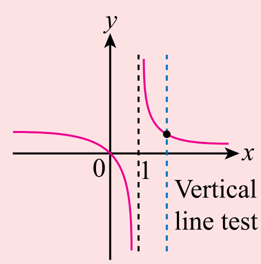


(c)

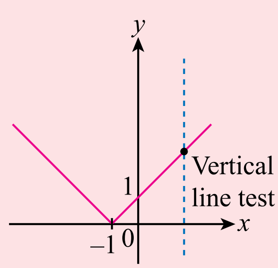


Solution

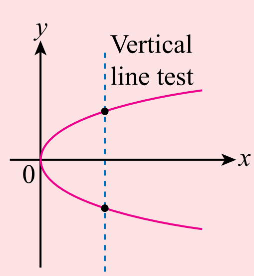
(a) This graph is a function because when the vertical line test is carried out, the line cuts the graph at only one point, except when $x = 1$ where the line does not cut any point on the graph.



(b) This graph is a function because when the vertical line test is carried out, the line cuts the graph at only one point.



(c) This graph is not a function because when the vertical line test is carried out, the line cuts the graph at two points.



Mind Challenge

How many x-intercepts and y-intercepts can exist in the graph of a function?

Observe the graph in Example 2(a). The graph is for the function $f(x) = \frac{x}{x-1}$. From the graph in Diagram 1.1, we obtain that when $x \to 1^-$, that is x approaches 1 from the left, $f(x) \to -\infty$, thus the value of $f(x)$ decreases non-terminating. When $x \to 1^+$, that is x approaches 1 from the right, $f(x) \to \infty$, thus the value of $f(x)$ increases non-terminating. This implies that the graph will approach but will never touch the line $x = 1$. Therefore, this function is not defined at $x = 1$.

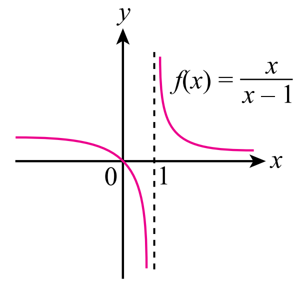


Diagram 1.1

Next, observe the graph in Example 2(b). The graph represents the function of absolute value $f(x) = |x + 1|$. The expression of the absolute value $|x|$ is the numerical value of x and is defined by:

$$|x| = \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$$

Thus, when $x = -2$, $|-2| = -(-2)$
$= 2$

and when $x = 2$, $|2| = 2$.

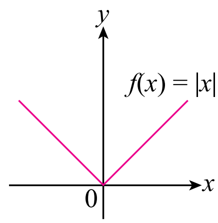


Diagram 1.2

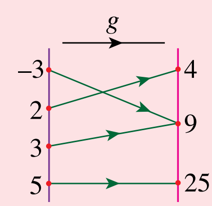
The function which is defined by $f(x) = |x|$ has a V-shaped graph where the vertex is at (0, 0) as shown in Diagram 1.2. $|x|$ is read as "the modulus of x".

MATHEMATICS POCKET

Based on Diagram 1.1:
- $x \to 1^-$ means x approaches 1 from the left side on the graph of $f(x) = \frac{x}{x-1}$, $x < 1$.
- $x \to 1^+$ means x approaches 1 from the right side on the graph of $f(x) = \frac{x}{x-1}$, $x > 1$.

Example 3

Based on the diagram on the right, write the relation for function g by using the function notation.



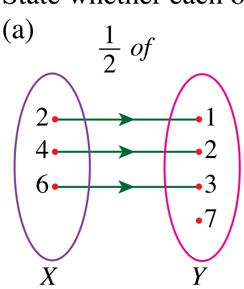
Solution

The function notation for the function is $g : x \to x^2$ or $g(x) = x^2$.

Self Practice 1.1

1. State whether each of the following relation is a function. State your reason.

(a)



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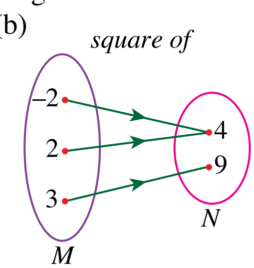

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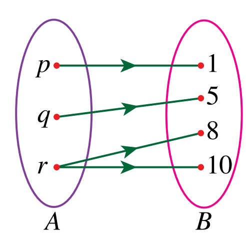
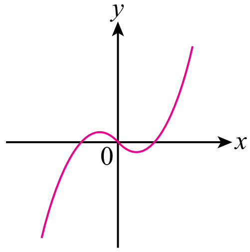


2. Determine whether each of the following graphs is a function by using the vertical line test.

(a)



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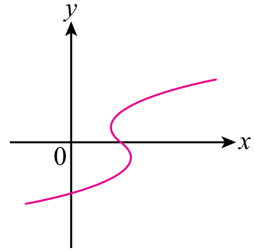
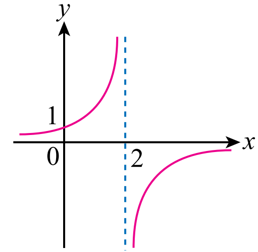
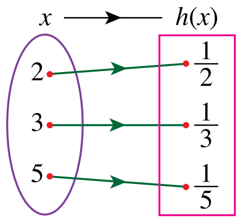


(c)

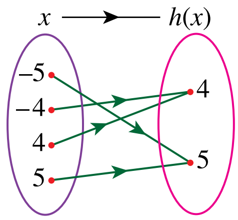


3. By using the function notation, express h in terms of x for each of the following arrow diagrams.

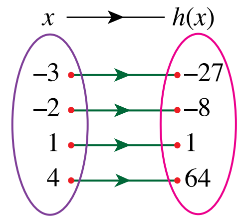
(a)



(b)



(c)



Determining the domain and range of a function

INQUIRY 1 In groups 21st Century Learning

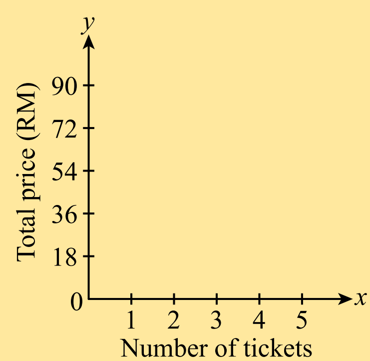
Aim: To explore the domain and range of a discrete function and a continuous function.

Instruction:

1. Each group is required to choose one of the situations below.

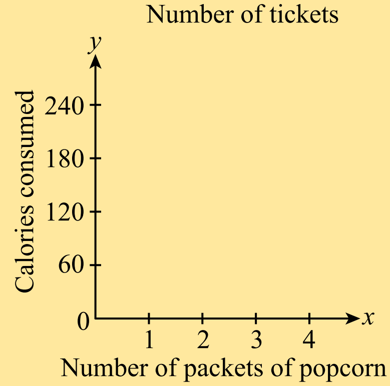
Situation I

Function $y = 18x$ represents the price of tickets, in RM, for x tickets purchased by a family to watch a film. Draw a graph of the function for the purchasing of 1 to 5 tickets.



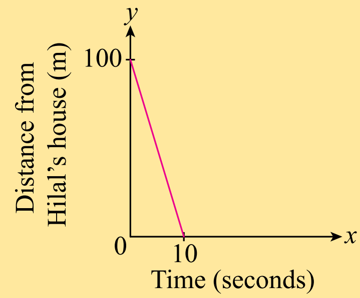
Situation II

A packet of popcorn contains 60 calories. y calories is a function of x packets of popcorn consumed. Draw a graph of the function for the purchasing of 1 to 4 packets of popcorn.



Situation III

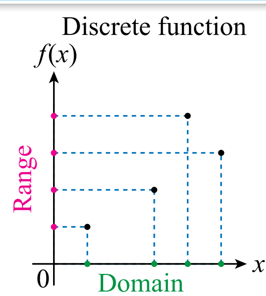
Hilal cycled a distance of 100 m from his friend's house at the speed of 10 ms^{-1}. With the same speed, Hilal cycled back to his friend's house to take his books that he left behind. Complete the distance-time graph of Hilal's journey.

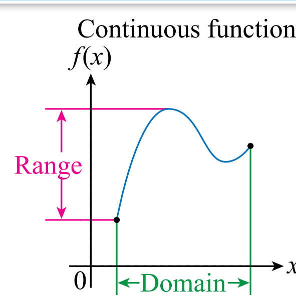


2. Based on the graph drawn, discuss with the members of your group and answer the following questions.
   (a) Is the graph of the function chosen discrete or continuous? Explain.
   (b) Identify the domain and range of the graph of function.
3. Present the findings of your group to the class.

From the results of Inquiry 1, it is noticed that points on the graph of the discrete function are real, separated and not connected by a straight line or a curve. As for the graphs of continuous function, the points are connected by a straight line or a curve within the given interval. Thus, Situation I represents a discrete function whereas Situations II and III represent continuous functions.

In general, the domain of a function is the set of possible values of x which defines a function, whereas range is the set of values of y that are obtained by substituting all the possible values of x.





Look at the arrow diagram of a discrete function f in Diagram 1.3. In this function, the elements in set X are mapped to a corresponding element in set Y respectively.

The elements in set X, are the values of x that can be substituted in f which is known as the **domain** whereas elements of set Y, are the possible obtained values of function f which is known as the **codomain**. The elements in set Y that are mapped from X, are the actual obtained values of function f is known as the **range**.

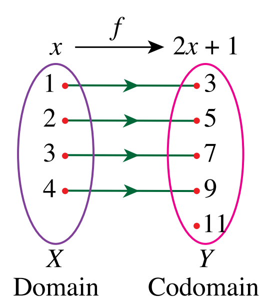


Diagram 1.3

Thus, we obtain

Domain = {1, 2, 3, 4}

Codomain = {3, 5, 7, 9, 11}

Range = {3, 5, 7, 9}

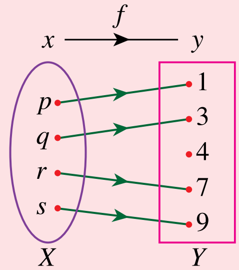
Next, consider a continuous function $f(x) = 2x + 1$ that can take all values of x from 1 to 4. Can you determine the domain, codomain and the range?

1.1.2

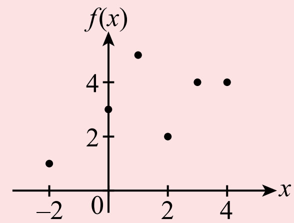
Example 4

Determine the domain, codomain and range for each of the following functions f.

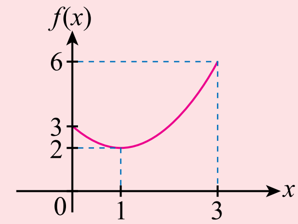
(a)



(b)



(c)



Solution

(a) Domain = $\{p, q, r, s\}$
Codomain = $\{1, 3, 4, 7, 9\}$
Range = $\{1, 3, 7, 9\}$

(b) Domain = $\{-2, 0, 1, 2, 3, 4\}$
Codomain = $\{1, 2, 3, 4, 5\}$
Range = $\{1, 2, 3, 4, 5\}$

(c) Domain of f is $0 \leqslant x \leqslant 3$.
Codomain of f is $2 \leqslant f(x) \leqslant 6$.
Range of f is $2 \leqslant f(x) \leqslant 6$.

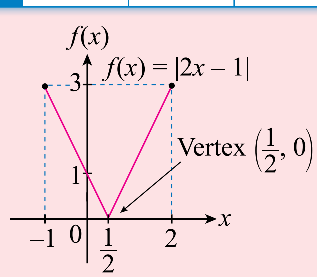
Example 5

Function f is defined as $f : x \rightarrow |2x - 1|$. Sketch the graph of f in the domain $-1 \leqslant x \leqslant 2$ and state the corresponding range of f for the given domain.

Solution

The graph of $f(x) = |2x - 1|$ can be sketched by plotting several points in the domain $-1 \leqslant x \leqslant 2$ as shown in the following table.

x	-1	0	$\frac{1}{2}$	1	2
$y = f(x) = \|2x - 1\|$	3	1	0	1	3
(x, y)	$(-1, 3)$	$(0, 1)$	$\left(\frac{1}{2}, 0\right)$	$(1, 1)$	$(2, 3)$



From the graph, the range of $f : x \rightarrow |2x - 1|$ is $0 \leqslant f(x) \leqslant 3$.

Alternative Method

From Example 5, draw the graph of $y = 2x - 1$ in the domain $-1 \leqslant x \leqslant 2$ first. The graph below the x-axis is reflected on the x-axis to obtain the graph of $f(x) = |2x - 1|$.

Tech Whizz

By using the *GeoGebra* software, draw the graphs of $y = |x|$, $y = 2|x|$, $y = 4|x|$ and $y = \frac{1}{2}|x|$.

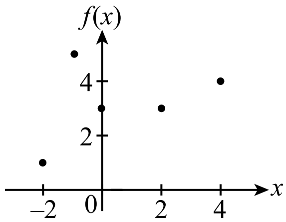
What pattern do you observe? Can you predict the graphs of $y = 8|x|$ and $y = \frac{1}{4}|x|$?

1.1.2

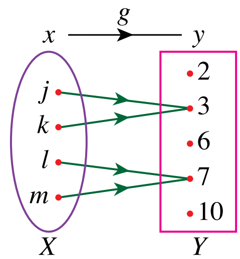
Self Practice 1.2

1. Determine the domain, codomain and range of the following functions.

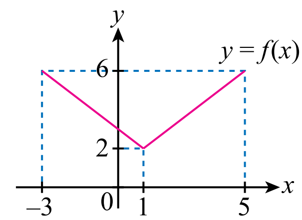
(a)



(b)



(c)



2. Sketch the graph of each of the following functions in the domain $-2 \leqslant x \leqslant 4$. Hence, state the corresponding range for the given domain.

(a) $f : x \rightarrow |x + 1|$ (b) $f(x) = |4 - 2x|$ (c) $f : x \rightarrow |2x - 5|$

Determining the image of a function when the object is given and vice versa

Consider a fruit juicer. When we put oranges into the juicer, orange juice will be obtained. It is impossible for us to obtain other than orange juice.

Think of this analogy where a function is the machine with input and output or an object and its image. Thus, if the object x is given and by inserting it in a function, the corresponding image $f(x)$ can be determined. Similarly, if the image, $f(x)$ is given, the corresponding object x can also be determined.

Example 6

Function f is defined by $f : x \rightarrow 3x + \dfrac{5}{x}, x \neq 0$. Find

(a) $f(5)$,

(b) the image of $\dfrac{1}{3}$ under f,

(c) the possible values of x when their image is 8.

Mind Challenge

$f : x \rightarrow 3x + \dfrac{5}{x}, x \neq 0.$

Why is $x \neq 0$?

If $f(x) = \dfrac{2}{x + 3}, x \neq k,$ what is the value of k?

Solution

(a) $f(5) = 3(5) + \dfrac{5}{5}$

$= 15 + 1$

$= 16$

(b) Given $f(x) = 3x + \dfrac{5}{x}$.

The image of $\dfrac{1}{3}$,

$f\left(\dfrac{1}{3}\right) = 3\left(\dfrac{1}{3}\right) + \dfrac{5}{\left(\dfrac{1}{3}\right)}$

$= 1 + 15$

$= 16$

(c) $f(x) = 8$

$3x + \dfrac{5}{x} = 8$ ← Multiply both sides of the equation with x.

$3x^2 + 5 = 8x$

$3x^2 - 8x + 5 = 0$

$(3x - 5)(x - 1) = 0$

$x = \dfrac{5}{3}$ or $x = 1$

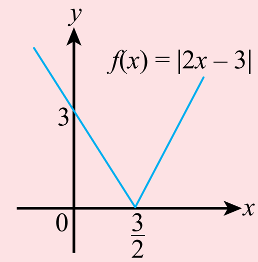
Thus, the possible values of x are $x = \dfrac{5}{3}$ and $x = 1$.

1.1.2 1.1.3

Example 7

The diagram on the right shows parts of the graph $f(x) = |2x - 3|$, find

(a) the values of $f(-2)$ and $f(4)$,
(b) the values of x such that $f(x) = 5$,
(c) the values of x which maps to itself,
(d) the domain of $f(x) < 1$,
(e) the domain of $f(x) \geqslant 3$.



Solution

(a)
$$\begin{aligned} f(-2) &= |2(-2) - 3| \\ &= |-7| \\ &= 7 \end{aligned}$$
$$\begin{aligned} f(4) &= |2(4) - 3| \\ &= |5| \\ &= 5 \end{aligned}$$

(b)
$$f(x) = 5$$
$$|2x - 3| = 5$$
$$\begin{aligned} 2x - 3 &= -5 \quad \text{or} \quad & 2x - 3 &= 5 \\ 2x &= -2 & 2x &= 8 \\ x &= -1 & x &= 4 \end{aligned}$$

(c)
$$f(x) = x$$
$$|2x - 3| = x$$
$$\begin{aligned} 2x - 3 &= -x \quad \text{or} \quad & 2x - 3 &= x \\ 3x &= 3 & x &= 3 \\ x &= 1 & & \end{aligned}$$

(d)
$$\begin{aligned} f(x) &< 1 \\ |2x - 3| &< 1 \\ -1 < 2x &- 3 < 1 \\ 2 < 2x &< 4 \\ 1 < x &< 2 \end{aligned}$$

(e) $f(x) \geqslant 3$
$$\begin{aligned} 2x - 3 &\leqslant -3 \quad \text{or} \quad & 2x - 3 &\geqslant 3 \\ 2x &\leqslant 0 & 2x &\geqslant 6 \\ x &\leqslant 0 & x &\geqslant 3 \end{aligned}$$

QR

The solution of equality and inequality involving absolute values.

bit.ly/2Oz1EcZ

Self Practice 1.3

1. Function g is defined by $g : x \rightarrow 3 + \dfrac{6}{x - 1}$, $x \neq 1$.
(a) Find the images of -5, -2 and $\dfrac{1}{2}$.
(b) Given the image of b is $2b$, find the possible values of b.

2. Function h is defined by $h : x \rightarrow \dfrac{kx - 3}{x - 1}$, $x \neq 1$. Find the value of k such that
(a) $h(2) = 5$ (b) $h(3) = k$ (c) $h(k) = k$

3. Function f is defined by $f : x \rightarrow |4x - 3|$, calculate
(a) $f(-2)$ and $f\left(-\dfrac{1}{2}\right)$, (b) the values of x such that $f(x) = 1$,
(c) the domain of $f(x) < 1$, (d) the domain of $f(x) > 5$.

4. Given $g(x) = |6 - 2x|$, find the values of x if $g(x) = x$.

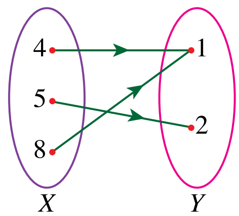
5. Function f is defined by $f : x \rightarrow mx + c$. Given $f(2) = 7$ and $f(4) = -1$, find
(a) the value of m and of c, (b) the image of 2 under f,
(c) the value of x that is unchanged under the mapping of f.

Intensive Practice 1.1

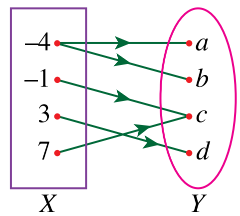
Scan the QR code or visit bit.ly/33iJznC for the quiz

1. Which of the following relations are functions? State your reasons.

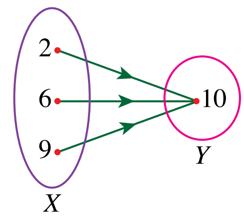
(a)



(b)

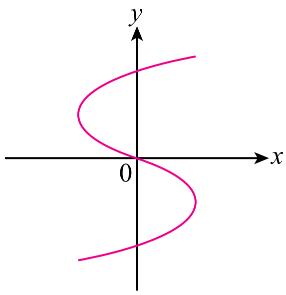


(c)

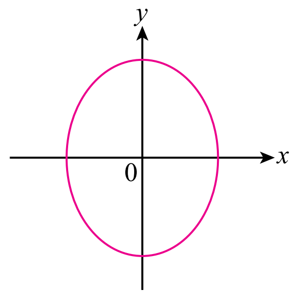


2. By using the vertical line test, determine whether the following graphs are functions.

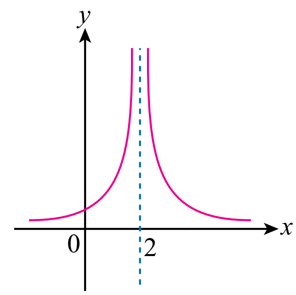
(a)



(b)

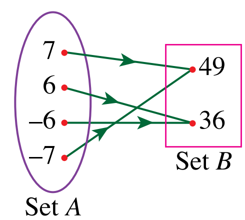


(c)



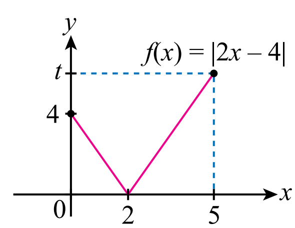
3. The diagram on the right shows the images for certain elements of set A.

(a) Is the relation a function? If so, state your reason.
(b) State the domain and range for that relation.
(c) Using the function notation, write one relation between set A and set B.



4. The diagram on the right shows the graph of the function $f(x) = |2x - 4|$ the domain $0 \leqslant x \leqslant 5$. Find

(a) the value of t,
(b) the range of f based on the given domain,
(c) the range of values of x such that $f(x) \leqslant 4$.



5. A stone fell from a height of 81 metres above the ground. The height of the stone, H metres, after t seconds, is assumed to be $H(t) = 81 - 9t^2$.

(a) State the height of the stone when

(i) $t = \frac{1}{3}$ second,
(ii) $t = 1$ second,
(iii) $t = 2$ seconds.

(b) When will the stone hit the ground?

1.2 Composite Functions

The picture on the right shows an oil leakage from a ship. The oil leakage forms a circle. The circular area of the oil leakage, A, is a function of its radius, r, in metres, and can be modelled as $A = f(r) = \pi r^2$.

The length of the radius, r increases with time, t, in hours, measured from the moment the leakage starts. This relationship can be modelled as $r = g(t) = 100t$. By substituting $r = 100t$ into the function $A = f(r) = \pi r^2$, we obtain:

$$\begin{aligned} A &= f(100t) \\ &= \pi(100t)^2 \\ &= 10\,000\ \pi t^2\ \text{m}^2 \end{aligned}$$

If the time t is given, then the area of the oil leakage can be determined. What can you say about the combination of the two functions $A = f(r)$ and $r = g(t)$ which results in $A = f[g(t)]$?

Describing the outcome of composition of two functions

INQUIRY 2 In groups | 21st Century Learning

Aim: To explore the outcome of composition of two functions f and g

Instruction:

1. Scan the *QR* code or visit the link on the right.
2. Given the functions $f(x) = x + 2$ and $g(x) = x^2$ together with the respective graphs.
3. Examine the graphs formed on the plane.
4. Click on the $f[g(x)]$ button and observe the graph displayed on the plane.
5. How to obtain the function $f[g(x)]$?
6. What is the shape of the graph resulted from the composition of the functions f and g?
7. Then, click again on the $f[g(x)]$ button to delete the graph of $f[g(x)]$.
8. Click on the $g[f(x)]$ button and observe the graph displayed on the plane.
9. How to obtain the graph of $g[f(x)]$?
10. What is the shape of the graph resulted from the composition of the functions g and f?
11. Then, change the functions f and g each with a different function to continue exploring the results of the composition of two functions and their graphs.
12. Each group will move to the other groups to see the results.
13. Discuss with the members of your group regarding the results obtained by the others.

bit.ly/2U5VrEq

From the results of Inquiry 2, it was found that the function $f[g(x)]$ is obtained by substituting the function g into the function f whereas the function $g[f(x)]$ is obtained by substituting the function f into the function g.

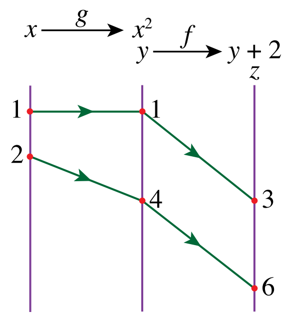
1.2.1

The process of combination by replacing two functions f and g to generate $f[g(x)]$ or $g[f(x)]$ is known as the composition of two functions and is written as $fg(x)$ or $gf(x)$.
$fg(x)$ is read as "f composed with g of x" and is defined by $fg(x) = f[g(x)]$.

> Given two functions $f(x)$ and $g(x)$, the product of combination of two functions that written as $fg(x)$ or $gf(x)$ are defined by $fg(x) = f[g(x)]$ or $gf(x) = g[f(x)]$.

Determining composite functions

Given functions $f(x) = x + 2$ and $g(x) = x^2$. The diagram below shows part of the mapping of function g followed by function f.



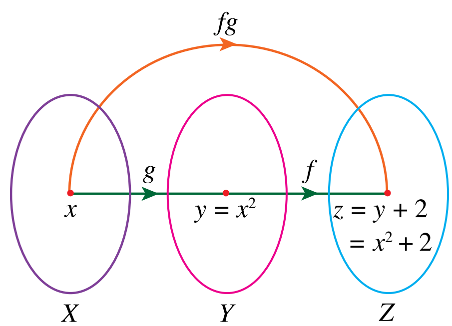
$1 \xrightarrow{g} 1^2 = 1 \xrightarrow{f} 1 + 2 = 3$

$2 \xrightarrow{g} 2^2 = 4 \xrightarrow{f} 4 + 2 = 6$

$x \xrightarrow{g} x^2 = y \xrightarrow{f} y + 2 = z = x^2 + 2$

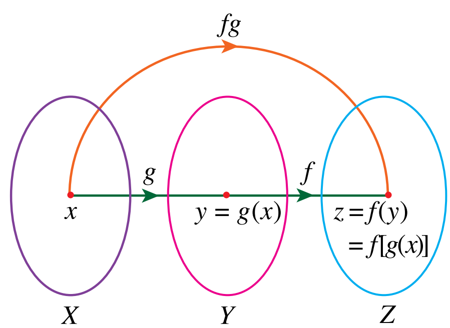
Based on the pattern in the diagram above, we can simplify it into an arrow diagram as shown on the right.

From the arrow diagram, a direct mapping can be seen with an element $x \in X$ mapped to an element $z \in Z$ which can be defined by the function $fg(x) = x^2 + 2$.



The new function of combining two functions f and g with domain X and codomain Z is known as the composite function of f and g which is represented by a function fg.

Thus, from the process shown, we can conclude that:



$$fg(x) = f[g(x)]$$

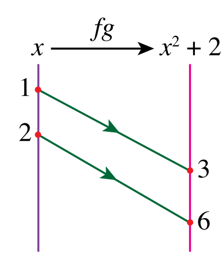
Algebraically, the composite function $fg(x)$ can be determined as follows:

$f(x) = x + 2$

$fg(x) = f[g(x)]$ ← $g(x) = x^2$

$\quad = f(x^2)$

$\quad = x^2 + 2$ or $fg : x \to x^2 + 2$



Example 8

Two functions are defined by $f : x \to 2x$ and $g : x \to x^2 - 5$.
Determine the following composite functions.

(a) fg (b) gf
(c) f^2 (d) g^2

Solution

(a) $fg(x) = f[g(x)]$
$= f(x^2 - 5)$
$= 2(x^2 - 5)$
$= 2x^2 - 10$
Thus, $fg : x \to 2x^2 - 10$

(b) $gf(x) = g[f(x)]$
$= g(2x)$
$= (2x)^2 - 5$
$= 4x^2 - 5$
Thus, $gf : x \to 4x^2 - 5$

(c) $f^2(x) = f[f(x)]$
$= f(2x)$
$= 2(2x)$
$= 4x$
Thus, $f^2 : x \to 4x$

(d) $g^2 = g[g(x)]$
$= g(x^2 - 5)$
$= (x^2 - 5)^2 - 5$
$= x^4 - 10x^2 + 25 - 5$
$= x^4 - 10x^2 + 20$
Thus, $g^2 : x \to x^4 - 10x^2 + 20$

Mind Challenge

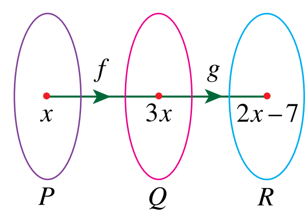
Would the composite functions, *fg* and *gf* always be different?

MATHEMATICS POCKET

f^2 is equal to ff.
Similarly, g^2 is equal to gg.

Self Practice 1.4

1. In the arrow diagram on the right, function f maps set P to set Q and function g maps set Q to set R.
Determine
(a) function f,
(b) function gf.



2. For each pair of the functions below, obtain an expression in the form of function notation for fg, gf, f^2 and g^2.
(a) $f : x \to 3x$, $g : x \to 3 - x$
(b) $f : x \to 4 + 2x$, $g : x \to x^2$
(c) $f : x \to x + 4$, $g : x \to \frac{6}{x}$, $x \neq 0$
(d) $f : x \to x - 5$, $g : x \to \frac{1}{x - 1}$, $x \neq 1$

3. Two functions, f and g are defined by $f : x \to 3x + 4$ and $g : x \to x^2 + 6$. Find the expressions for fg and gf, then find the values of x when
(a) $f = g$
(b) $fg = gf$

4. Given that $f : x \to ax + b$ and $f^2 : x \to 4x - 9$, find the value of the constants a and b.

5. If $f : x \to 3x + k$ and $g : x \to 2h - 3x$ such that $fg = gf$, find the relation between h and k.

Determining the image or object of a composite function

By substituting the value of the object into a composite function, the image can be obtained. Similarly, if the value of the image is given, then the value of the object can be determined by solving the equation.

Example 9

If $f : x \rightarrow x - 1$ and $g : x \rightarrow x^2 - 3x + 4$, find
(a) $fg(2)$ and $gf(1)$,
(b) the values of x when $fg(x) = 7$.

Solution

(a)
$$\begin{aligned} fg(x) &= f[g(x)] \\ &= f(x^2 - 3x + 4) \\ &= x^2 - 3x + 4 - 1 \\ &= x^2 - 3x + 3 \end{aligned}$$
Thus, $fg(2) = (2)^2 - 3(2) + 3$
$= 1$
$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g(x - 1) \\ &= (x - 1)^2 - 3(x - 1) + 4 \\ &= x^2 - 2x + 1 - 3x + 3 + 4 \\ &= x^2 - 5x + 8 \end{aligned}$$
Thus, $gf(1) = (1)^2 - 5(1) + 8$
$= 4$

(b)
$$\begin{aligned} fg(x) &= 7 \\ x^2 - 3x + 3 &= 7 \\ x^2 - 3x - 4 &= 0 \\ (x + 1)(x - 4) &= 0 \\ x = -1 &\text{ or } x = 4 \end{aligned}$$
Thus, the values of x are -1 and 4.

Alternative Method

(a) $g(2) = 2^2 - 3(2) + 4$
$= 2$
Thus, $fg(2) = f(2)$
$= 2 - 1$
$= 1$

Self Practice 1.5

1. Given two functions f and g.
   (a) $f : x \rightarrow 2x + 1$ and $g : x \rightarrow \frac{x}{x - 1}, x \neq 1$, find $fg(3)$.
   (b) $f : x \rightarrow 5x + 6$ and $g : x \rightarrow 2x - 1$, find $gf\left(-\frac{1}{5}\right)$.
   (c) $f : x \rightarrow \frac{x + 1}{x - 3}, x \neq 3$ and $g : x \rightarrow \frac{6}{x - 2}, x \neq 2$, find $f^2(4)$ and $g^2\left(\frac{1}{2}\right)$.
   (d) $f : x \rightarrow x^2 - 4$ and $g : x \rightarrow \frac{2}{x - 2}, x \neq 2$, find $f^2(-1)$ and $g^2(1)$.
2. For each of the following function, find the value of the object x.
   (a) $f : x \rightarrow 2x - 5$, $g : x \rightarrow \frac{10}{x}, x \neq 0$ and $fg(x) = 5$.
   (b) $f : x \rightarrow x^2 - 1$, $g : x \rightarrow 2x + 1$ and $gf(x) = 7$.
   (c) $f : x \rightarrow 3x - 2$ and $f^2(x) = 10$.
   (d) $g : x \rightarrow \frac{2}{x - 2}, x \neq 2$ and $g^2(x) = -\frac{1}{2}$.

1.2.3

Determining a function when the composite function and one of the functions are given

When the composite function and one of the functions are given, the other function can be determined.

Example 10

The function f is defined by $f : x \rightarrow x - 2$. Find the function g for each of the following.

(a) $fg : x \rightarrow 8x - 7$

(b) $gf : x \rightarrow x^2 + 3x - 5$

Solution

(a) $f[g(x)] = 8x - 7$

$g(x) - 2 = 8x - 7$

$g(x) = 8x - 7 + 2$

$g(x) = 8x - 5$

Thus, $g : x \rightarrow 8x - 5$

(b) $g[f(x)] = x^2 + 3x - 5$

$g(x - 2) = x^2 + 3x - 5$

Suppose $y = x - 2$

$x = y + 2$

Then, $g(y) = (y + 2)^2 + 3(y + 2) - 5$

$= y^2 + 4y + 4 + 3y + 6 - 5$

$= y^2 + 7y + 5$

Replacing y with x, $g(x) = x^2 + 7x + 5$

Thus, $g : x \rightarrow x^2 + 7x + 5$

Self Practice 1.6

1. Given the function f and the composite function fg, determine the function g for each of the following.

(a) $f : x \rightarrow x - 3, fg : x \rightarrow 2x^2 - 4x + 7$

(b) $f : x \rightarrow x^2 + 1, fg : x \rightarrow x^2 + 4x + 5$

2. Given the function f and the composite function gf, determine the function g for each of the following.

(a) $f : x \rightarrow x + 1, gf : x \rightarrow x^2 - 2x - 3$

(b) $f : x \rightarrow x^2 + 3, gf : x \rightarrow 2x^2 + 3$

3. Given the function $h(x) = \frac{8}{x}, x \neq 0$ and $hg(x) = 4x$, find

(a) $g(x)$,

(b) the value of x when $gh(x) = 6$.

4. Given the function $g(x) = 3x$ and $fg(x) = 9x - 7$, find

(a) $f(x)$,

(b) $gf(2)$.

Solving problems involving composite functions

Example 11

Function f is defined by $f : x \rightarrow \frac{1}{x^2}, x \neq 0$.

(a) Express $f^2(x), f^3(x)$ and $f^4(x)$ in the simplest form.

(b) Hence, find $f^{22}(x)$ and $f^{33}(x)$.

Solution

(a) $f(x) = \frac{1}{x^2} = x^{-2}$

$f^2(x) = f[f(x)]$
$= f\left(\frac{1}{x^2}\right)$
$= \frac{1}{\left(\frac{1}{x^2}\right)^2}$
$= x^4$
$= x^{2^2}$

$f^3(x) = f[f^2(x)]$
$= f(x^4)$
$= \frac{1}{(x^4)^2}$
$= \frac{1}{x^8}$
$= x^{-2^3}$

$f^4(x) = f[f^3(x)]$
$= f\left(\frac{1}{x^8}\right)$
$= \frac{1}{\left(\frac{1}{x^8}\right)^2}$
$= x^{16}$
$= x^{2^4}$

(b) From the pattern in (a), we can deduce that $f^n(x) = x^{-2^n}$ when n is odd and $f^n(x) = x^{2^n}$ when n is even. Thus, $f^{22}(x) = x^{2^{22}}$ and $f^{33}(x) = x^{-2^{33}}$.

Example 12 MATHEMATICS APPLICATION

Total production of q goods per day by a factory depends on the number of workers, n, and the function is modelled by $q(n) = 10n - \frac{1}{4}n^2$. Total revenue per day, r, in RM, received from the sale of q goods is modelled by the function $r(q) = 40q$. Determine the total revenue of the factory in one day if the number of workers is 20.

Solution

1. Understanding the problem

- Given two functions, q and r which are defined by $q(n) = 10n - \frac{1}{4}n^2$ and $r(q) = 40q$ respectively.
- Find the total revenue of the factory of 20 workers in one day.

2. Planning the strategy

- Find the composite function $rq(n)$ so as to determine the total revenue of the factory, r which represents the function of n workers, that is $r(n)$.
- Substitute $n = 20$ into the composite function $r(n)$ which is obtained earlier so as to find the total revenue of the factory per day, in RM.

1.2.5

3. Implementing the strategy

$rq(n) = r[q(n)]$

$= r\left(10n - \frac{1}{4}n^2\right)$

$= 40\left(10n - \frac{1}{4}n^2\right)$

$= 400n - 10n^2$

Therefore, $r(n) = 400n - 10n^2$

With 20 workers,

$r(20) = 400(20) - 10(20^2)$

$= 8\ 000 - 4\ 000$

$= 4\ 000$

Thus, the daily revenue of the factory with 20 workers is RM4 000.

4. Making a conclusion

When $r(n) = 4\ 000$,

$4\ 000 = 400n - 10n^2$

$10n^2 - 400n + 4\ 000 = 0$

$n^2 - 40n + 400 = 0$

$(n - 20)(n - 20) = 0$

$n = 20$

Thus, the daily revenue of the factory is RM4 000 when the number of workers is 20.

Self Practice 1.7

1. The function f is defined by $f : x \rightarrow \frac{x}{x+1}, x \neq -1$.

(a) Find the iterated functions f^2, f^3 and f^4.

(b) Hence, write the functions f^{20} and f^{23}.

2. If $f : x \rightarrow \frac{1}{x}, x \neq 0$, find

(a) the iterated functions f^2, f^3 and f^4,

(b) values of $f^{40}(2)$ and $f^{43}(2)$.

3. The surface area of a hot air balloon, A, in m^2, filled with hot air is given by the function $A(r) = 4\pi r^2$ where r is the radius of the balloon, in metres. The radius of the balloon is increasing as a function of time, t, in seconds, according to the formula $r(t) = \frac{2}{3}t^3, t \geqslant 0$.

(a) State the surface area of the balloon, A, as a function of time, t.

(b) Find the surface area of the balloon after 2 seconds.

4. A cylindrical container of radius 20 cm contains 200 cm^3 of water. The container is filled with water at a constant rate of 100 cm^3 per second.

(a) Write the formula of

(i) the amount of water in the container, v, after t seconds,

(ii) the height of water in the container, h, in terms of v,

(iii) the composite function $hv(t)$.

(b) Find the height of water in the container after 20 seconds.

5. A small stone is thrown into a calm pond and produces a circular ripple. The radius of the ripple, r, in cm, is increasing at a rate of 3 cm per second.

(a) Find an expression for the radius, r, in terms of time, t, after the stone is thrown.

(b) If A is the area of the ripple, explain the meaning of the composite function $Ar(t)$.

(c) Find the area A, of the ripple after 30 seconds.

Intensive Practice 1.2

Scan the QR code or visit bit.ly/2AWCFIn for the quiz

1. Two functions are defined by $f : x \to 2x - 1$ and $g : x \to \frac{x}{x+1}, x \neq -1$. Find

(a) fg and gf,　(b) $fg(2)$ and $gf\left(-\frac{1}{2}\right)$,　(c) the value of x when $fg = gf$.

2. The functions f and g are defined by $f : x \to \frac{x}{x-1}, x \neq 1$ and $g : x \to hx + k$, where h and k are constants. Given $g(3) = 8$ and $gf(2) = 5$, find

(a) the value of h and of k,　(b) the value of a if $fg(a) = 3$.

3. The functions f and g are defined by $f : x \to ax - b$ where a and b are constants and $g : x \to x + 4$. Given $fg(2) = 9$ and $gf\left(\frac{1}{2}\right) = 2$, find the value of a and of b.

4. The functions f and g are defined by $f : x \to \frac{2}{x-3}, x \neq 3$ and $g : x \to hx^2 + k$, where h and k are constants.

(a) Given $g(2) = 5$ and $gf(1) = -1$, calculate the value of h and of k.

(b) Find the expression of gf.

5. Given that $f : x \to ax + b$ and $f^3 : x \to 27x + 13$, find

(a) the value of a and of b,

(b) the expression of f^4.

6. The diagram on the right shows a tissue box with a square base of side x cm and the height of 10 cm.

(a) Write the base area of the box, A as a function of x and its volume, V as a function of A.

(b) Show that the volume, V is the result of the composition of these two functions.

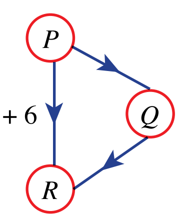
7. The function f is defined by $f : x \to x + 6$. Find the function g in each of the following.

(a) $fg : x \to 2x^2 - 3x - 7$　(b) $gf : x \to x^2 + 4$　(c) $gf : x \to 8 - x$

8. The diagram on the right shows the relation between set P, set Q and set R. Given that set P maps to set Q by the function $\frac{x-1}{3}$ and maps to set R by $fg : x \to x^2 - 3x + 6$.

(a) Write the function that maps set P to set Q by using the function notation.

(b) Find the function that maps set Q to set R.



9. Given $f : x \to px + q$ and $f^3 : x \to 8x - 7$,

(a) find the value of p and of q,

(b) determine the function f^4,

(c) by studying the pattern of f, f^2, f^3 and f^4, determine the general rule f^n where n is the number of times.

10. A car factory manufactures N cars daily after t hours of operation is given by $N(t) = 100t - 5t^2$, $0 \leqslant t \leqslant 10$. If the cost, in RM, for manufacturing x cars is $C(N) = 15\,000 + 8\,000x$, find the cost C as a function of time t, for the operation of the factory.

1.3 Inverse Functions

You read news online that the temperature in New York is 39°F. Calculate the temperature in degree Celsius.

The relationship between the number on a Fahrenheit, F thermometer and that of degree Celsius, C is a function $F(C) = \frac{9}{5}C + 32$. By changing C as the subject of the formula, that is $C(F) = \frac{5}{9}(F - 32)$ and substituting the value $F = 39$ into the function C, the temperature in degree Celsius of New York can be known.

(*Source: https://www.necn.com/weather/maps/ NECN-Weather-Now-250228521.html*)

$$F : C \rightarrow \frac{9}{5}C + 32$$
$$C : F \rightarrow \frac{5}{9}(F - 32)$$

Will carrying out the inverse operation as shown above generate an inverse function of F?

Describing the inverse of a function

Inverse function of a function f can be written as f^{-1}. For example:

$f : x \rightarrow x + 2$

$f^{-1} : x \rightarrow x - 2$

What is an inverse of a function? To understand this further, let's follow the next exploration.

INQUIRY 3 In groups 21st Century Learning

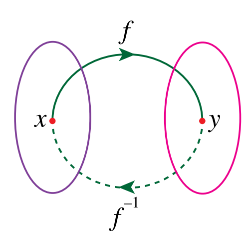
Aim: To explore the relation between the graph of a function and its respective inverse.

Instruction:

1. Scan the QR code or visit the link on the right.
2. Click on the buttons of all the functions and study the graphs obtained.
3. Are the graphs of each function and its inverse symmetrical about the line $h(x) = x$?
4. Carry out the discussion in the respective groups.

bit.ly/2LQjG8M

From the result of Inquiry 3, it was found that every graph of the function and its graph of inverse function is symmetrical about the line $h(x) = x$, that is $y = x$. The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.



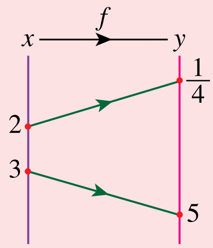
$$f : x \rightarrow y \Leftrightarrow f^{-1} : y \rightarrow x \text{ or } y = f(x) \Leftrightarrow x = f^{-1}(y)$$

1.3.1

Example 13

In the arrow diagram on the right, the function f maps x to y. Determine

(a) $f^{-1}\left(\frac{1}{4}\right)$ (b) $f^{-1}(5)$



Solution

(a) From the given arrow diagram, we obtain

$f(2) = \frac{1}{4}$, thus $f^{-1}\left(\frac{1}{4}\right) = 2$.

(b) By inverse mapping, $f^{-1} : 5 \rightarrow 3$.
Then, $f^{-1}(5) = 3$. ← $f : x \rightarrow y \Leftrightarrow f^{-1} : y \rightarrow x$

MATHEMATICS POCKET

The sign –1 used in f^{-1} does not mean the reciprocal of f, $f^{-1}(x) \neq \frac{1}{f(x)}$ but f^{-1} is the inverse of f.

Example 14

A function is defined as $f(x) = \frac{x}{x-4}, x \neq 4$. Determine

(a) the image of 2 under f, (b) $f^{-1}(3)$.

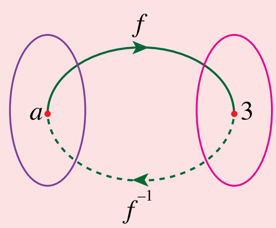
Solution

(a) The image of 2, $f(2) = \frac{2}{2-4} = -1$

(b) Let $a = f^{-1}(3)$,

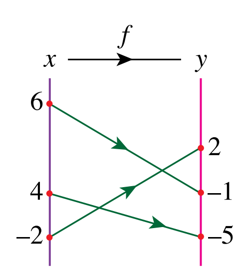
$$f(a) = 3$$
$$\frac{a}{a-4} = 3$$
$$a = 3(a-4)$$
$$a = 3a - 12$$
$$2a = 12$$
$$a = 6$$

Thus, $f^{-1}(3) = a = 6$



Self Practice 1.8

1. In the arrow diagram on the right, the function f maps x to y. Find
   (a) $f(4)$
   (b) $f^{-1}(-1)$
   (c) $f^{-1}(2)$
   (d) $f^{-1}(-5)$



2. The functions g and h are defined by $g(x) = \frac{5}{2-x}, x \neq 2$ and $h(x) = 3x + 6$ respectively, find

   (a) $g(12)$ (b) $g^{-1}(4)$ (c) $h(-1)$ (d) $h^{-1}(9)$

1.3.1

Making and verifying conjectures related to the properties of inverse functions

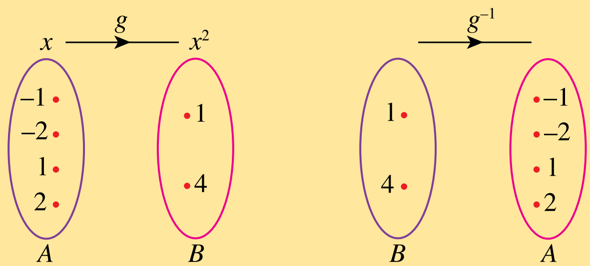
Do the following Inquiry 4, 5, 6 and 7 to make and verify the conjectures regarding the properties of inverse functions.

INQUIRY 4 In groups 21st Century Learning

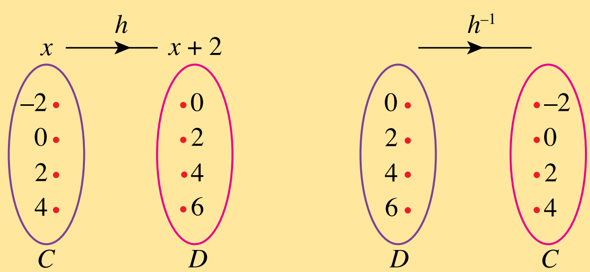
Aim: To make and verify conjectures that a one-to-one function has an inverse function

Instruction:

1. Copy and complete the mapping of the following discrete functions.
   (a) Discrete function g maps set A to set B and g^{-1} maps set B to set A.



   (b) The discrete function h maps set C to set D and h^{-1} maps set D to set C.



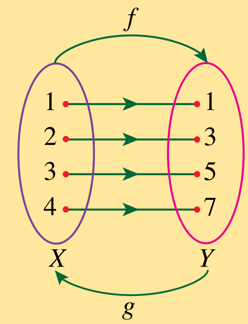
2. Are g^{-1} and h^{-1} functions?
3. What type of function will yield an inverse function? State your conjecture.
4. Every group must send a representative to present the outcome of the group's findings in front of the class. Members of the other groups can ask the representative questions.
5. Repeat step 4 until all the groups have completed the presentation.

From the results of Inquiry 4, it was found that the inverse function is the reversal of a function such that every element in the codomain is mapped to only one element in the domain. Thus, we can conclude that:

> A function f that maps set X to set Y has an inverse function, f^{-1} if f is a one-to-one function.

INQUIRY 5 In groups 21st Century Learning

Aim: To make and verify conjectures that composite function $fg(x) = gf(x) = x$ such that function f and function g are inverses of each other



Instruction:

1. The arrow diagram on the right shows the discrete function f that maps set X to set Y and the discrete function g that maps set Y to set X.
2. Complete the empty boxes below based on the arrow diagram on the right.

$f(1) = 1$
$f(2) = 3$
$f(3) = \square$
$f(\square) = \square$

$g(1) = 1$
$g(3) = 2$
$g(5) = \square$
$g(\square) = \square$

$gf(1) = g(1) = 1$
$fg(1) = f(1) = 1$

$gf(2) = g(3) = 2$
$fg(3) = f(2) = 3$

$gf(3) = g(\square) = \square$
$fg(5) = f(\square) = \square$

$gf(4) = g(\square) = \square$
$fg(7) = f(\square) = \square$

3. From the result obtained, what is the conjecture that can be made regarding the values of $fg(x)$ and $gf(x)$?
4. Each group presents their findings in front of the class and a question and answer session is carried out.

From the results of Inquiry 5, the functions f and g are the inverse functions of each other if and only if:

$$fg(x) = x, x \text{ in the domain of } g \text{ and } gf(x) = x, x \text{ in the domain of } f.$$

INQUIRY 6 In groups 21st Century Learning

Aim: To make and verify conjectures that if two functions f and g are inverse functions of each other, then
(a) domain of f = range of g and domain of g = range of f
(b) graph of g is the reflection of graph of f at the line $y = x$

bit.ly/2ob10HU

Instruction:

1. Scan the QR code or visit the link on the right.
2. Click on the box $f(x) = \frac{1}{2}x$ for the domain $0 \leqslant x \leqslant 8$ and take note of the graph obtained.
3. Then, click on the box $g(x) = 2x$, that is the inverse of f and take note of the graph displayed.
4. Complete the domain and range of the graphs of f and g in the table shown on the right.

Graph	Domain	Range
Graph of function f		
Graph of function g		

5. What is your conjecture regarding the results of the findings?
6. How are the graphs of f and g located with respect to the line $y = x$? What is your conjecture?
7. Each representative of the groups moves to the other groups and presents the results of their group.

1.3.2

From the results of Inquiry 6, it was found that:

If two functions f and g are inverse functions of each other, then
(a) the domain of f = range of g and domain of g = range of f
(b) the graph of g is the reflection of the graph of f at the line $y = x$

INQUIRY 7 In groups 21st Century Learning

Aim: To make and verify conjecture that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of g

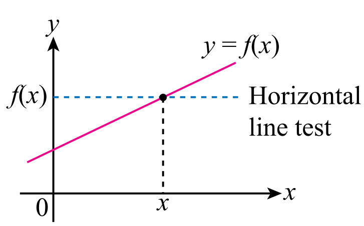
bit.ly/30PXAr9

Instruction:
1. Scan the *QR* code or visit the link on the right.
2. Click on the box $f(x) = x^2 + 1$ in the range $0 \leqslant x \leqslant 3$ and its inverse function, $g(x) = \sqrt{x - 1}$ in the range $1 \leqslant x \leqslant 10$.
3. Then, click on the box "Point and reflection". Drag point A along the graph f. Note the points on the graph f and the graph g.
4. What is the conjecture that can be made regarding the points that you observe on both graphs?
5. Carry out a discussion within the group regarding the results of the finding.
6. Each group elects a representative and presents in front of the class.

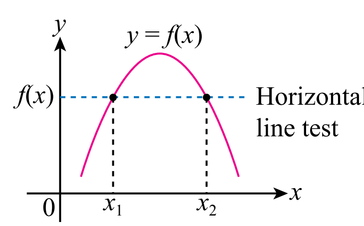
From the results of Inquiry 7, it was found that:

For any real number, a and b, if the point (a, b) lies on the graph f, then the point (b, a) lies on the graph g, that is graph f^{-1}. The point (b, a) lies on the graph g is the point of reflection of (a, b) which lies on the graph f in the line $y = x$.

To determine whether the graph of a function has an inverse function, carry out the horizontal line test. If the horizontal line cuts the graph of the function at only one point, then this type of function is a one-to-one function and it has an inverse function. Conversely, if the horizontal line cuts the graph at two or more points, then this type of function is not a one-to-one and the function has no inverse function.



f has an inverse function

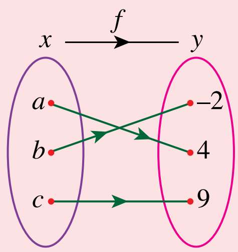


f does not have an inverse function

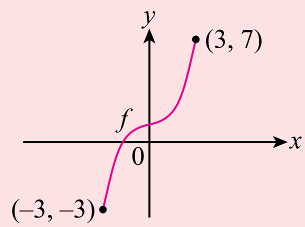
Example 15

Determine whether each of the following functions f has an inverse or not. Give a reason for your answer.

(a)

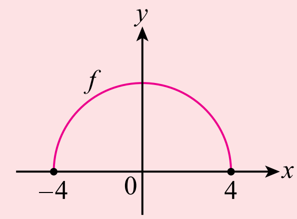


(b)



Domain: $-3 \leqslant x \leqslant 3$

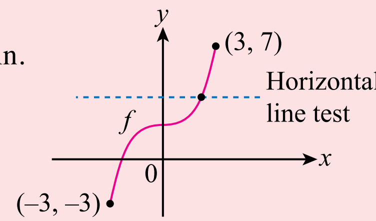
(c)



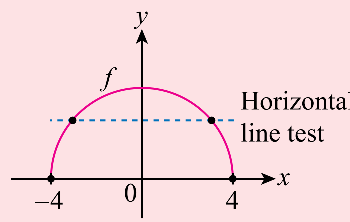
Domain: $-4 \leqslant x \leqslant 4$

Solution

(a) f is a function because this function is one-to-one with every element in the domain mapped to only one element in the codomain. The inverse of this function also maps every element in the codomain to only one element in the domain. Thus, function f has an inverse function.



(b) When the horizontal line test is carried out, the horizontal line cuts the graph of function f at only one point. This means that function f is a one-to-one function. Thus, function f has an inverse function.



(c) When the horizontal line test is carried out, the horizontal line cuts the graph of function f at two points. This means that function f is not a one-to-one function. Thus, function f has no inverse function.

Example 16

Verify that the function $f(x) = 3 - 2x$ has an inverse function, $g(x) = \dfrac{3 - x}{2}$.

Solution

First, determine $fg(x)$.

$$\begin{aligned} fg(x) &= f[g(x)] \\ &= f\left(\frac{3 - x}{2}\right) \\ &= 3 - 2\left(\frac{3 - x}{2}\right) \\ &= 3 - (3 - x) \\ &= x \end{aligned}$$

Then, determine $gf(x)$.

$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g(3 - 2x) \\ &= \frac{3 - (3 - 2x)}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

Since $fg(x) = gf(x) = x$, thus $g(x) = \dfrac{3 - x}{2}$ is the inverse function of $f(x) = 3 - 2x$.

1.3.2

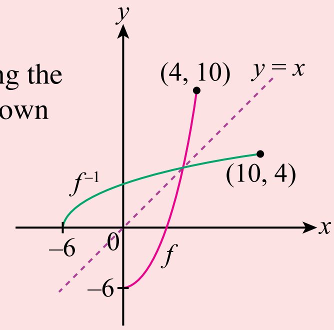
Example 17

The function f is defined by $f : x \rightarrow x^2 - 6$ for the domain $0 \leqslant x \leqslant 4$. On the same plane, sketch the graphs of f and f^{-1}. Hence, state the domain of f^{-1}.

Solution

The graph of f is a part of the quadratic curve $y = x^2 - 6$. By plotting the points in the table of values below, the graph of f is sketched as shown in the diagram on the right.

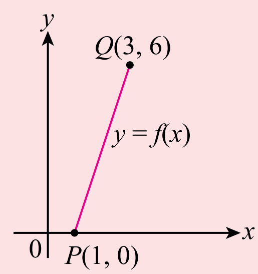
x	0	1	2	3	4
y	–6	–5	–2	3	10



The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.
The domain of f^{-1} is the range of f. Hence, the domain of f^{-1} is $-6 \leqslant x \leqslant 10$.

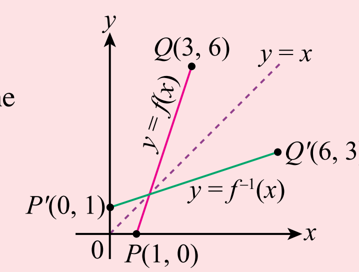
Example 18

The diagram on the right shows the graph of $y = f(x)$ passing through the points $P(1, 0)$ and $Q(3, 6)$. On the same diagram, sketch the graph of $y = f^{-1}(x)$ by showing the points corresponding to point P and point Q.



Solution

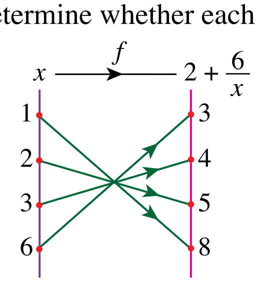
The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$. The points P' and Q' on the graph of $y = f^{-1}(x)$ correspond to the points P and Q as shown in the diagram on the right.



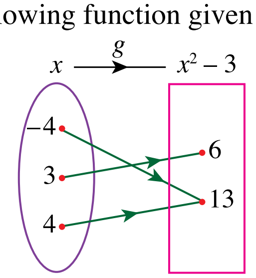
Self Practice 1.9

1. Determine whether each of the following function given has an inverse function.

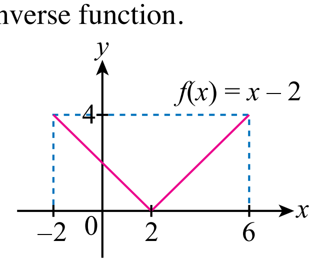
(a)



(b)



(c)



(d) $\{(1, 2), (4, 5), (5, 8), (9, 9)\}$

(e) $\{(-3, 2), (-1, 1), (2, 4), (5, 4), (9, 5)\}$

(f) $f : x \rightarrow 4 - x^2$

(g) $f : x \rightarrow \dfrac{1}{(x - 2)^2}, x > 2$

2. Are the following functions f and g the inverse functions of each other? Verify the truth by applying the relation $fg(x) = gf(x) = x$.

(a) $f(x) = 3x - 2$ and $g(x) = \dfrac{x+2}{3}$

(b) $f(x) = \dfrac{2x}{x-3}, x \neq 3$ and $g(x) = \dfrac{3x}{x-2}, x \neq 2$

(c) $f(x) = \dfrac{2}{x-3}, x \neq 3$ and $g(x) = \dfrac{3x-2}{x}, x \neq 0$

(d) $f(x) = 2 + 5x$ and $g(x) = \dfrac{x-5}{2}$

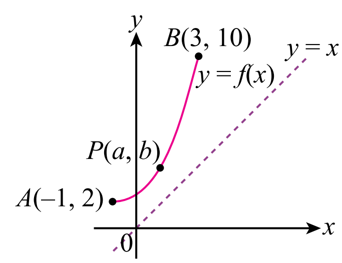
3. The function f is defined as $f : x \rightarrow x^3$ for the domain $-1 \leqslant x \leqslant 2$. On the same plane, sketch the graphs of f and f^{-1}. Hence, state the domain and range of f^{-1}.

4. The function h is defined as $h(x) = x^2 - 2$ for the domain $0 \leqslant x \leqslant 3$.
(a) On the same diagram, sketch the graphs of h and h^{-1}.
(b) State the domain of h^{-1}.
(c) Find the value of x such that $h(x) = h^{-1}(x)$.

5. The coordinates of the following points lie on the graph of one-to-one function, f. Determine the corresponding coordinates lying on the graph of f^{-1}.

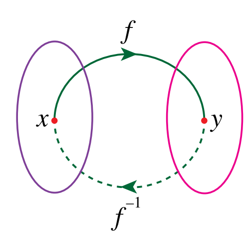
(a) $P\left(-2, \dfrac{1}{2}\right)$ (b) $Q(1, -3)$ (c) $R(4, 5)$ (d) $S(-6, -8)$

6. The diagram on the right shows the line $y = x$ and the graph of $y = f(x)$ for the domain $-1 \leqslant x \leqslant 3$. The points $A(-1, 2)$, $B(3, 10)$ and $P(a, b)$ lie on the graph.
(a) Sketch the graph of $y = f^{-1}(x)$ that shows the points on the graph of $y = f^{-1}(x)$ corresponding to the points A and B.
(b) Find the values of a and b, if the corresponding coordinates on the graph of $y = f^{-1}(x)$ are $(4, 1)$.



Determining the inverse function

We have learnt that when given $y = f(x)$, then $x = f^{-1}(y)$. Algebraically, the formula of the inverse function, $f^{-1}(x)$ with the original function $y = f(x)$ can be determined by following the steps below.



Change function $y = f(x)$ to the form $x = f(y)$. → Write x as $f^{-1}(y)$. → Replace variable y with variable x.

Example 19

If $f : x \to 5x + 2$, find
(a) $f^{-1}(x)$ (b) $f^{-1}(7)$

Solution

(a) $f(x) = 5x + 2$

Let $y = 5x + 2$

$5x = y - 2$

$x = \dfrac{y-2}{5}$ ← The form of $x = f(y)$

Since $x = f^{-1}(y)$,

$f^{-1}(y) = x$ ← Write x as $f^{-1}(y)$

$= \dfrac{y-2}{5}$

Substitute the variable y with x,

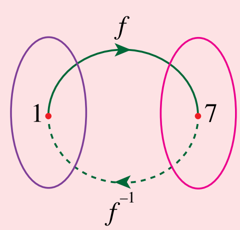
$f^{-1}(x) = \dfrac{x-2}{5}$

Thus, $f^{-1} : x \to \dfrac{x-2}{5}$.

(b) $f^{-1}(x) = \dfrac{x-2}{5}$

Thus, $f^{-1}(7) = \dfrac{7-2}{5}$

$= 1$



MATHEMATICS POCKET

Check the validity of the inverse function $f^{-1}(x) = \dfrac{x-2}{5}$ obtained in Example 19(a) by applying the relationship $ff^{-1}(x) = f^{-1}f(x) = x$.

$ff^{-1}(x) = f[f^{-1}(x)]$

$= 5\left(\dfrac{x-2}{5}\right) + 2$

$= x$

$f^{-1}f(x) = f^{-1}[f(x)]$

$= f^{-1}(5x + 2)$

$= \dfrac{5x+2-2}{5}$

$= x$

Since $ff^{-1}(x) = f^{-1}f(x) = x$, thus $f^{-1} : x \to \dfrac{x-2}{5}$ is the inverse function of $f : x \to 5x + 2$.

Self Practice 1.10

1. Find f^{-1} for each of the following one-to-one functions.

(a) $f : x \to 2x - 5$ (b) $f : x \to \dfrac{3}{x}, x \neq 0$ (c) $f : x \to \dfrac{4}{x-1}, x \neq 1$

(d) $f : x \to \dfrac{5x}{x-6}, x \neq 6$ (e) $f : x \to \dfrac{x+9}{x-8}, x \neq 8$ (f) $f : x \to \dfrac{2x-3}{2x-1}, x \neq \dfrac{1}{2}$

2. The function f is defined by $f : x \to \dfrac{3-x}{2x}, x \neq 0$, find

(a) $f^{-1}(4)$,

(b) the values of x such that $f(x) = f^{-1}(x)$.

3. Given the functions $h : x \to 4x + a$ and $h^{-1} : x \to 2bx + \dfrac{5}{8}$, find the value of the constants a and b.

4. Find the function f in similar form for each of the following f^{-1}.

(a) $f^{-1} : x \to 6x + 7$ (b) $f^{-1} : x \to \dfrac{2-x}{5}$ (c) $f^{-1} : x \to \dfrac{3x}{x-3}, x \neq 3$

5. The inverse function g^{-1} is defined by $g^{-1} : x \to \dfrac{4}{2-x}, x \neq k$.

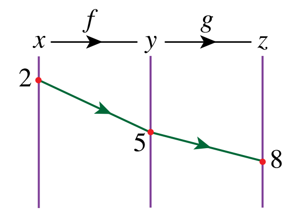
(a) State the value of k. (b) Find $g\left(\dfrac{1}{2}\right)$.

Intensive Practice 1.3

Scan the QR code or visit bit.ly/2p5bB7a for the quiz

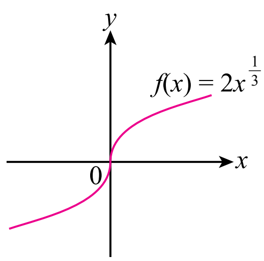
1. In the arrow diagram on the right, the function f maps x to y and the function g maps y to z. Determine

(a) $f(2)$ (b) $g(5)$
(c) $gf(2)$ (d) $f^{-1}(5)$
(e) $g^{-1}(8)$ (f) $f^{-1}g^{-1}(8)$

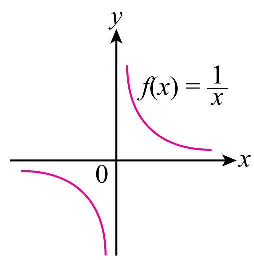


2. By applying the horizontal line test, determine whether each of the following function has an inverse function.

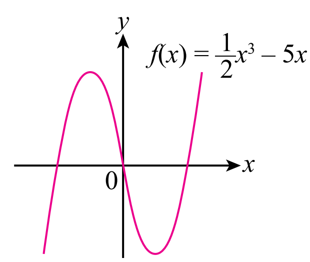
(a)



(b)

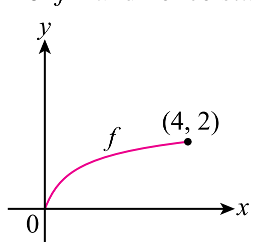


(c)

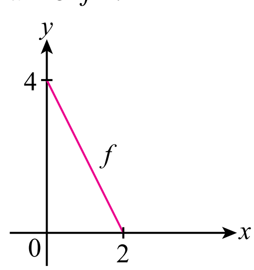


3. The diagrams below show the graphs of one-to-one functions, f. In each case, sketch the graph of f^{-1} and hence state the domain of f^{-1}.

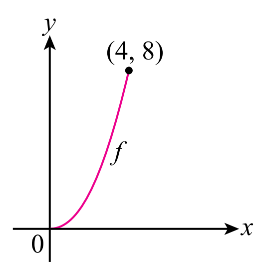
(a)



(b)



(c)



4. Given $f : x \to \dfrac{2x + h}{x - 3}$, $x \neq 3$ and $f(4) = 13$, find

(a) the value of h, (b) $f^{-1}(3)$,
(c) the value of m when $f^{-1}(m) = 2$.

5. The inverse function h^{-1} is defined by $h^{-1} : x \to \dfrac{2}{3 - x}$, $x \neq 3$, find

(a) $h(x)$, (b) the values of x such that $h(x) = 2$.

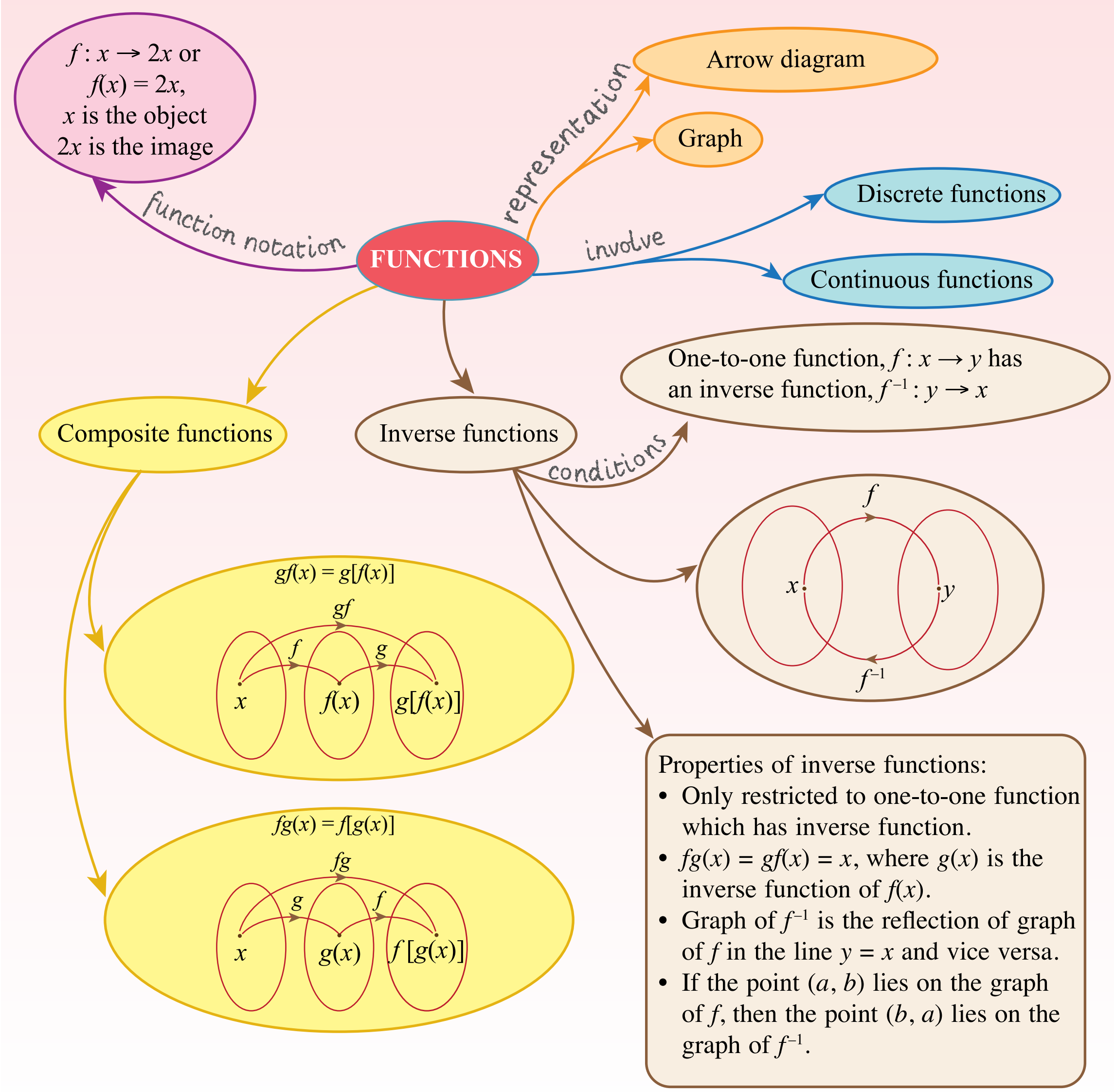
6. Two functions f and g are defined by $f : x \to 4x - 17$ and $g : x \to \dfrac{5}{2x - 7}$, $x \neq 3\frac{1}{2}$. Solve the equation $f^{-1}(x) = g^{-1}(x)$.

7. Faridah carried out a physical activity during her leisure time. Then, Faridah calculated her estimated heartbeat rate by using the function $f(x) = \dfrac{17}{20}(220 - x)$, where x is her age.

(a) Determine the inverse function of this function.
(b) If Faridah's age is 16, determine her estimated heartbeat rate.

8. Zaki intends to make spherical water balls that can hold $\frac{1}{2}$ cm^3 of water. The volume of sphere, V is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. Zaki wishes to know how to determine r if V is given.
(a) Draw an arrow diagram of the function f that maps r to V and its inverse function f^{-1} which maps V to r.
(b) Hence, determine the radius of the ball that can hold the volume of water according to the specification.

SUMMARY OF CHAPTER 1



WRITE YOUR JOURNAL

Search on the internet or books regarding the history of the usage of the function notation $y = f(x)$. Create a digital folio by using presentation software such as *PowerPoint*, *Prezi or Powtoon*.

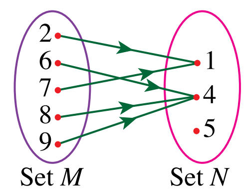
MASTERY PRACTICE

1. The arrow diagram on the right shows the relationship between set M and set N. PL1

(a) State
   (i) the image of 2,
   (ii) the object of 4.

(b) Does this relation represent a function? Give your reason.

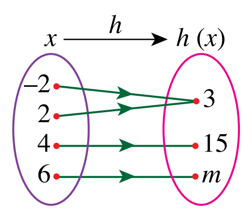
(c) State the domain, codomain and the range of the relation.



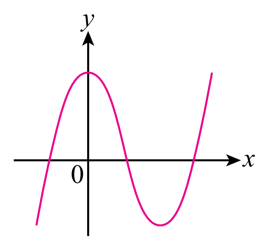
2. The arrow diagram on the right shows a function h. PL2

(a) State the value of m.

(b) Using the function notation, express h in terms of x.



3. Using the vertical line test, determine whether the graph on the right represents a function or not. If yes, is this function a one-to-one function? Test by drawing a horizontal line on the graph. PL2



4. Function f is defined by $f : x \rightarrow |x - 3|$ for the domain $-1 \leqslant x \leqslant 7$. PL3

(a) Sketch the graph of f and state the range of f.

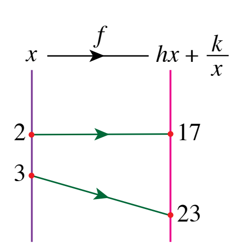
(b) Find the range of values of x such that $f(x) \leqslant 2$.

(c) On the same graph in part (a), sketch the graph of $y = 2x - 3$ and hence obtain the value of x such that $|x - 3| = 2x - 3$.

5. The arrow diagram on the right represents part of the mapping of $f : x \rightarrow hx + \dfrac{k}{x}, x \neq 0$, find PL3

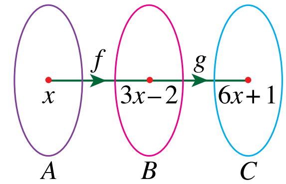
(a) the value of h and of k,

(b) the image of 6 under this mapping.



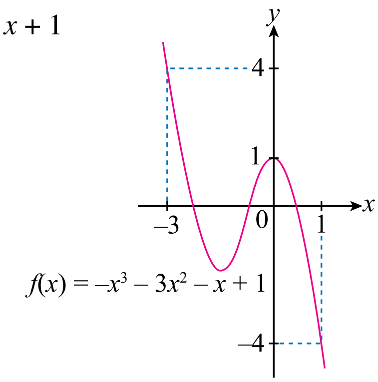
6. Two functions f and g are defined by $f : x \rightarrow \dfrac{x+2}{x-2}, x \neq 2$ and $g : x \rightarrow mx + c$. Given that $g^{-1}(2) = f(3)$ and $gf^{-1}(2) = 5$, find the value of m and of c. PL3

7. In the diagram on the right, the function f maps set A to set B and the function g maps set B to set C. Find PL4
(a) in terms of x, the function
(i) that maps set B to set A,
(ii) $g(x)$.
(b) the value of x such that $fg(x) = 4x - 3$.



8. Function f is defined by $f : x \rightarrow \dfrac{m}{x-1} + n, x \neq k$. Given $f(2) = 3$ and $f(3) = 2$, find PL3
(a) the value of k,
(b) the value of m and of n,
(c) $f^2(x)$,
(d) $f^{-1}(2)$.

9. The diagram on the right shows the function $f(x) = -x^3 - 3x^2 - x + 1$ for the domain $-3 \leqslant x \leqslant 1$. PL3
(a) State
(i) whether the function f is discrete or continuous,
(ii) the range of values of f corresponding to the domain given.
(b) By applying the horizontal line test, determine whether f has an inverse function or not.



10. Given that the functions $f(x) = |x|$ and $f(x) = x^4$ are not one-to-one functions. PL5
(a) Determine the suitable conditions in the domain of f so that the new functions become one-to-one functions.
(b) From (a), find the inverse function for each of the functions f.

11. If the graphs of a function and its inverse function intersect, would the two graphs intersect on the line $y = x$? What is the possibility for the two graphs to intersect on other lines? PL5

12. Given $f(x) = \dfrac{ax+b}{cx+d}$, find $f^{-1}(x)$. PL5
(a) Using the formula f^{-1} obtained, determine f^{-1} for each of the following functions.
(i) $f(x) = \dfrac{x+8}{x-5}, x \neq 5$
(ii) $f(x) = \dfrac{2x-3}{x+4}, x \neq -4$
(b) If $c \neq 0$, what are the conditions on a, b, c and d so that $f = f^{-1}$?

13. A one-to-one function f is defined by $f : x \rightarrow x^2 - 2x$ for $1 \leqslant x \leqslant 3$. PL6
(a) Using the *GeoGebra* software,
(i) draw the graph of f and from the graph, state the range of f,
(ii) draw the graph of f^{-1} on the same plane and state the domain of f^{-1}.
(b) What can you say about the range of f and the domain of f^{-1} and also the domain of f and the range of f^{-1}? Hence, on the same plane, draw the line $y = x$.
(i) Is the graph of f^{-1} the reflection of the graph f in that line?
(ii) Is the point $(0, 2)$ on the graph of f^{-1} the reflection of the point $(2, 0)$ on the graph of f in the line $y = x$? What conclusion can you make?

14. The price p, in RM, of an item and the quantity x sold follow the demand equation $p = 100 - \frac{1}{4}x$ for $0 \leqslant x \leqslant 400$. Whereas the cost C, in RM, to produce x units is $C = \frac{\sqrt{x}}{25} + 600$. Assuming all the items produced are sold, calculate PL4
(a) the cost C as a function of price p,
(b) the cost for producing that item if the price for one unit of the item is sold at RM36.

15. Period T, in seconds, of a simple pendulum is a function of length l, in metres, defined by $T(l) = 2\pi\sqrt{\frac{l}{g}}$, such that $g = 10$ m s^{-2} is the gravitational acceleration. Using the *GeoGebra* software, draw the graph of this function and on the same plane, draw the graphs of the following functions.
(a) $T(l) = 2\pi\sqrt{\frac{l+4}{g}}$
(b) $T(l) = 2\pi\sqrt{\frac{4l}{g}}$

How does the change in the length, affect the period, T of the pendulum? PL5

Exploring MATHEMATICS

The table below shows the amount of petrol used by a car on a highway as compared to the distance travelled. Supposed l is the volume of petrol used, in litres, and d is the distance travelled, in km, by the car.

Petrol used (l)	Distance travelled, in km (d)
4	48
8	96
12	144
16	192
20	240

1. Based on the table above,
(a) how far can the car travel with 1 litre of petrol?
(b) determine the distance travelled, d, as a function of amount of petrol used, l.

$d(l)$ = ____________

2. Using the *GeoGebra* software, draw the function d as obtained in question 1(b) and from the graph, determine the following:
(a) What is the amount of petrol used to travel 300 km?
(b) What is the distance that can be travelled for 100 l of petrol?

CHAPTER 2

Quadratic Functions

What will be learnt?

- Quadratic Equations and Inequalities
- Types of Roots of Quadratic Equations
- Quadratic Functions

List of Learning Standards

bit.ly/2AYDj8t

KEYWORDS

Completing the square	*Penyempurnaan kuasa dua*
Root	*Punca*
General form	*Bentuk am*
Quadratic inequality	*Ketaksamaan kuadratik*
Number line	*Garis nombor*
Discriminant	*Pembezalayan*
Real root	*Punca nyata*
Imaginary root	*Punca khayalan*
Vertex form	*Bentuk verteks*
Axis of symmetry	*Paksi simetri*
Maximum value	*Nilai maksimum*
Minimum value	*Nilai minimum*

The cross section of a skateboard ramp in the shape of a parabola can be modelled by using the quadratic function, that is $f(x) = ax^2 + bx + c$. For your knowledge, the shape and width of a skateboard ramp can be modified through the knowledge of quadratic functions. What is the best shape of skateboard ramp from the safety aspects?

Did you Know?

A satellite dish has the ability to converge the energy on its focal point. Satellite, television, radar and telecommunication tower are examples of objects which focus on the properties of reflection of parabola.

Based on the history of the ancient times, Archimedes helped the Greek army by using parabolic mirrors to torch the military ships of Rome who were trying to conquer the Greek town, Syracuse in 213 B.C.

For further information:

bit.ly/35rNxMi

SIGNIFICANCE OF THIS CHAPTER

- Astronomers use the concept of quadratic function in inventing telescopes. Parabolic mirrors are able to converge and reflect the light onto a certain point.
- In the engineering field, engineers apply the concept of quadratic function to determine the types of loads which can be accommodated by a bridge.

Scan this QR code to watch a video on skateboard games in Malaysia.

bit.ly/2V2I1ys

2.1 Quadratic Equations and Inequalities

Solving quadratic equations by using the method of completing the square and quadratic formula

Most of the situations that take place in our daily lives are associated with equations. One of the equations is the quadratic equation. Consider this situation:

> The area of a rectangular picture frame is 100 cm^2. If the length is 3 cm longer than its width, write an equation which satisfies this situation.

Supposed that the width of the frame is x cm and its length is 3 cm longer than its width, that is $(x + 3)$ cm. Then:

$$x(x + 3) = 100$$
$$x^2 + 3x = 100$$
$$x^2 + 3x - 100 = 0$$

Note that this equation has a variable x and the highest power of the variable is 2. Hence, this equation is known as a quadratic equation in general form. In general, a general form of quadratic equation can be written as:

> $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$.

How do you solve a quadratic equation? What does it mean by solving the quadratic equation?

INQUIRY 1 In pairs | 21st Century Learning

Aim: To explore the solving of quadratic equations by using dynamic geometry software

Instructions:

1. Scan the *QR* code or visit the link on the right.
2. Click the *point* button and mark *A* and *B* on the intersection points between the graph $y = 3x^2 + 11x - 4$ and the x-axis.
3. Record the coordinates of point *A* and point *B*. Then, observe the x-coordinate of point *A* and point *B*.
4. What is the conclusion that can be made on the x-coordinate of point *A* and point *B*?
5. Discuss with your partner and share the findings obtained with other classmates.

bit.ly/30MWj48

From the results of Inquiry 1, the values of x for both the intersection points, that is, $x = -4$ and $x = \frac{1}{3}$ are the solutions or roots of the equation $y = 3x^2 + 11x - 4$ when $y = 0$.

Then, it can be concluded that:

Solutions or roots of the quadratic equation $ax^2 + bx + c = 0$ are the x-coordinates of the intersection points between the graph $y = ax^2 + bx + c$ and the x-axis.

You have learned how to solve quadratic equations by using factorisation method. Other than that, the solutions of a quadratic equation can be obtained by using the **completing the square** and **formula** method.

A Completing the square method

Example 1

Solve the following equations by using completing the square method.

(a) $x^2 + 4x - 7 = 0$

(b) $-3x^2 + 6x - 1 = 0$

Solution

(a)

$$x^2 + 4x - 7 = 0$$

$$x^2 + 4x = 7$$

Move the constant term to the right hand side of the equation

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = 7 + \left(\frac{4}{2}\right)^2$$

Add the term $\left(\frac{\text{coefficient of } x}{2}\right)^2$ on the left and right hand side of the equation

$$x^2 + 4x + 2^2 = 7 + 2^2$$

$$(x + 2)^2 = 11$$

$(x + a)^2 = x^2 + 2ax + a^2$

$$x + 2 = \pm\sqrt{11}$$

$$x = -5.317 \quad \text{or} \quad x = 1.317$$

Hence, the solutions of equation $x^2 + 4x - 7 = 0$ are -5.317 and 1.317.

(b)

$$-3x^2 + 6x - 1 = 0$$

$$x^2 - 2x + \frac{1}{3} = 0$$

Divide both sides of the equation by -3 so that the coefficient of x^2 becomes 1

$$x^2 - 2x = -\frac{1}{3}$$

$$x^2 - 2x + \left(\frac{-2}{2}\right)^2 = -\frac{1}{3} + \left(\frac{-2}{2}\right)^2$$

Add $\left(\frac{-2}{2}\right)^2$ on both sides of the equation

$$x^2 - 2x + (-1)^2 = -\frac{1}{3} + (-1)^2$$

$$(x - 1)^2 = \frac{2}{3}$$

$$x - 1 = \pm\sqrt{\frac{2}{3}}$$

$$x = 0.1835 \quad \text{or} \quad x = 1.8165$$

Hence, the solutions of equation $-3x^2 + 6x - 1 = 0$ are 0.1835 and 1.8165.

FLASHBACK

Factorisation method

$x^2 + 5x + 6 = 0$

$(x + 2)(x + 3) = 0$

$x = -2$ or $x = -3$

QR

Factorisation method by using algebra tiles.

bit.ly/34f7x3G

Mathematics Museum

A Persian mathematician, Abu Ja'far Muhammad ibn Musa al-Khawarizmi used the same method as completing the square to solve quadratic equations.

2.1.1

B Formula method

The formula for solving a quadratic equation $ax^2 + bx + c = 0$ is given as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2

Solve the equation $2x^2 - 2x - 3 = 0$ by using formula.

Solution

Compare the given equation with the equation of general form $ax^2 + bx + c = 0$. Hence, $a = 2$, $b = -2$ and $c = -3$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{28}}{4}$$
$$x = \frac{2 - \sqrt{28}}{4} \quad \text{or} \quad x = \frac{2 + \sqrt{28}}{4}$$
$$= -0.823 \quad \text{or} \quad = 1.823$$

Hence, the solutions of the equation $2x^2 - 2x - 3 = 0$ are -0.823 and 1.823.

Mind Challenge

Derive the quadratic formula by using completing the square method.

BRAINSTORMING

State another method to solve a quadratic equation other than the method of completing the square and quadratic formula. What is your choice? Explain the reason for your choice.

Mathematics Museum

An Indian mathematician and astronomer, Brahmagupta produced a formula to solve a quadratic equation $ax^2 + bx + c = 0$ which is equivalent to $x = \frac{\sqrt{4ac + b^2} - b}{2a}$.

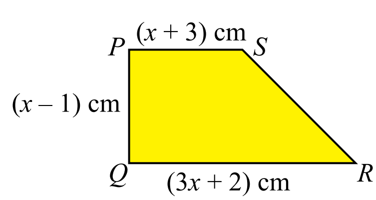
Self Practice 2.1

1. Solve the following quadratic equations by using completing the square method. Give your answers in three decimal places.
   (a) $x^2 + 4x - 9 = 0$ (b) $x^2 - 3x - 5 = 0$ (c) $-x^2 - 6x + 9 = 0$
   (d) $2x^2 - 6x + 3 = 0$ (e) $4x^2 - 8x + 1 = 0$ (f) $-2x^2 + 7x + 6 = 0$

2. Solve the following quadratic equations by using formula. Give your answers in three decimal places.
   (a) $x^2 - 4x - 7 = 0$ (b) $2x^2 + 2x - 1 = 0$ (c) $3x^2 - 8x + 1 = 0$
   (d) $4x^2 - 3x - 2 = 0$ (e) $(x - 1)(x - 3) = 5$ (f) $(2x - 3)^2 = 6$

3. (a) The length of the diagonal of a rectangle is 10 cm. If the length is 2 cm longer than its width, find the length and the width of the rectangle.
   (b) Find the measurements of a rectangle with a perimeter of 26 cm and an area of 40 cm^2.

4. The diagram on the right shows a trapezium $PQRS$ where $PQ = (x - 1)$ cm, $PS = (x + 3)$ cm and $QR = (3x + 2)$ cm. Given the area of the trapezium is 17 cm^2, find the value of x.



2.1.1

CHAPTER 2

Forming quadratic equations from given roots

The quadratic equation $ax^2 + bx + c = 0$ can be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \cdots ①$$

If α and β are the roots of a quadratic equation, then

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \cdots ②$$

Comparing ① and ②,

$$-(\alpha + \beta) = \frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$
$$\alpha + \beta = -\frac{b}{a}$$

In general, this comparison can be formulated as follows:

$$\text{Sum of roots} = \alpha + \beta = -\frac{b}{a}$$
$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

Therefore, the quadratic equation with roots α and β can be written as:

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

FLASHBACK

Factorisation identity
(a) $(x + y)^2 = (x + y)(x + y)$
$= x^2 + 2xy + y^2$
(b) $(x - y)^2 = (x - y)(x - y)$
$= x^2 - 2xy + y^2$
(c) $x^2 - y^2 = (x + y)(x - y)$

BRAINSTORMING

Given $\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$,
(a) show that
$\alpha + \beta = -\frac{b}{a}$,
(b) express the product of $\alpha\beta$ in terms of a and c.

Discuss with your classmates.

Example 3

Form a quadratic equation with roots 3 and –5.

Solution

Given $\alpha = 3$ dan $\beta = -5$.

Sum of roots, $\alpha + \beta = 3 + (-5)$
$= -2$

Product of roots, $\alpha\beta = 3 \times (-5)$
$= -15$

Thus, the quadratic equations with roots 3 and –5 is

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$
$$x^2 - (-2)x + (-15) = 0$$
$$x^2 + 2x - 15 = 0$$

Alternative Method

$$(x - 3)(x + 5) = 0$$
$$x^2 + 5x - 3x - 15 = 0$$
$$x^2 + 2x - 15 = 0$$

2.1.2

CHAPTER 2

Example 4

If α and β are the roots of the quadratic equation $2x^2 + x = 4$, form a quadratic equation with the following roots.

(a) $\alpha + 3, \beta + 3$

(b) $2\alpha, 2\beta$

(c) α^2, β^2

Solution

$2x^2 + x - 4 = 0$ where $a = 2, b = 1$ and $c = -4$

$\alpha + \beta = -\frac{b}{a} = -\frac{1}{2}$ and $\alpha\beta = \frac{c}{a} = -\frac{4}{2} = -2$

(a) Sum of roots:

$$(\alpha + 3) + (\beta + 3) = (\alpha + \beta) + 6 = -\frac{1}{2} + 6 = \frac{11}{2}$$

Product of roots:

$$(\alpha + 3)(\beta + 3) = \alpha\beta + 3(\alpha + \beta) + 9 = -2 + 3\left(-\frac{1}{2}\right) + 9 = \frac{11}{2}$$

Thus, the quadratic equation with roots $\alpha + 3$ and $\beta + 3$ is

$$x^2 - \frac{11}{2}x + \frac{11}{2} = 0$$

Multiply both sides of the equation by 2

$$2x^2 - 11x + 11 = 0$$

(b) Sum of roots:

$$2\alpha + 2\beta = 2(\alpha + \beta) = 2\left(-\frac{1}{2}\right) = -1$$

Product of roots:

$$(2\alpha)(2\beta) = 4\alpha\beta = 4(-2) = -8$$

Thus, the quadratic equation with roots 2α and 2β is

$$x^2 - (-1)x - 8 = 0$$
$$x^2 + x - 8 = 0$$

(c) Sum of roots:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{1}{2}\right)^2 - 2(-2) = \frac{1}{4} + 4 = \frac{17}{4}$$

Product of roots:

$$\alpha^2\beta^2 = (\alpha\beta)^2 = (-2)^2 = 4$$

Thus, the quadratic equation with roots α^2 and β^2 is

$$x^2 - \frac{17}{4}x + 4 = 0$$

Multiply both sides of the equation by 4

$$4x^2 - 17x + 16 = 0$$

Self Practice 2.2

1. Form quadratic equations which have the following roots.
   (a) 2 and 6 (b) −1 and 4 (c) −4 and −7 (d) $\frac{1}{5}$ and −5

2. The quadratic equation $x^2 + (p - 5)x + 2q = 0$ has roots of –3 and 6. Find the value of p and of q.

3. If α and β are the roots of the quadratic equation $5x^2 - 10x - 9 = 0$, form quadratic equations with the following roots.
   (a) $\alpha + 2$ and $\beta + 2$ (b) 5α and 5β (c) $\alpha - 1$ and $\beta - 1$ (d) $\frac{\alpha}{3}$ and $\frac{\beta}{3}$

4. If α and β are the roots of the quadratic equation $2x^2 + 5x = 1$, find the quadratic equations with the following roots.
   (a) $\frac{1}{\alpha}, \frac{1}{\beta}$ (b) $\left(\alpha + \frac{1}{\beta}\right), \left(\beta + \frac{1}{\alpha}\right)$ (c) α^2, β^2 (d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

5. A quadratic equation $2x^2 = 6x + 3$ has two roots p and q. Find the quadratic equations with roots p^2q and pq^2.

Solving quadratic inequalities

An inequality with a quadratic expression on one side and zero on the other side, is called a quadratic inequality in one variable. For example, $2x^2 + 7x - 4 \leqslant 0$ and $(x + 1)((x - 3) > 0$ are quadratic inequalities in one variable, x. To solve a quadratic inequality as $(x + 1)(x - 3) > 0$, we have to find the range of values of x so that the expression on the left is greater than zero.

The three methods which can be used to solve a quadratic inequality are **graph sketching**, **number line** and **table** methods.

INQUIRY 2 In groups 21st Century Learning

Aim: To solve quadratic inequalities by graph sketching, number line and table methods

Instructions:

1. Consider quadratic inqualities $(x + 1)(x - 3) > 0$ and $(x + 1)(x - 3) < 0$.
2. Form three groups and each group has to choose one of the three following method.

Graph sketching method

- Solve the quadratic equations $(x + 1)(x - 3) = 0$.
- Draw the graph of $y = (x + 1)(x - 3)$.
- Mark and determine the range of values of x on the graph when $(x + 1)(x - 3) > 0$ ($y > 0$) and $(x + 1)(x - 3) < 0$ ($y < 0$).

Number line method

- Solve the quadratic equation $(x + 1)(x - 3) = 0$.
- Draw a number line on a piece of paper.
- By choosing the values of x that satisfy $x < -1$, $x > 3$ and $-1 < x < 3$ on the number line and substituting them into $(x + 1)(x - 3)$, determine and verify the range of values of x when $(x + 1)(x - 3) > 0$ and $(x + 1)(x - 3) < 0$.

Table method

- Copy and complete the table with positive value (+) or negative value (–) for every factor of quadratic equation $(x + 1)(x - 3) = 0$.

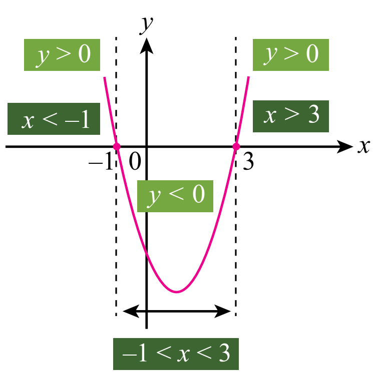
	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
$(x + 1)$					
$(x - 3)$					
$(x + 1)(x - 3)$					

- From the results obtained in the table, what is the range of values of x when $(x + 1)(x - 3) > 0$ and $(x + 1)(x - 3) < 0$?

3. Compare the findings of your group with other groups.
4. Do a thorough discussion about the three methods that can be used to solve quadratic equations.

From the results of Inquiry 2, the solution for the quadratic inequalities $(x + 1)(x - 3) > 0$ and $(x + 1)(x - 3) < 0$ by using the methods of graph sketching, number lines and table are shown as follow:

Graph sketching

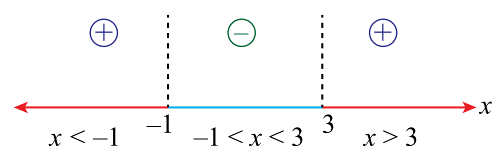


Number line

Test point –2: $(-2 + 1)(-2 - 3) > 0$

Test point 0: $(0 + 1)(0 - 3) < 0$

Test point 4: $(4 + 1)(4 - 3) > 0$



Table

	Range of values of x		
	$x < -1$	$-1 < x < 3$	$x > 3$
$(x + 1)$	–	+	+
$(x - 3)$	–	–	+
$(x + 1)(x - 3)$	+	–	+

From the three findings above, it can be concluded that:

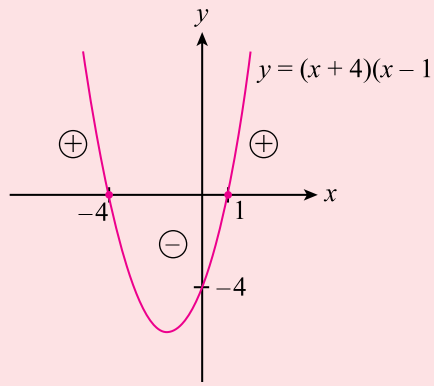
For a quadratic equation in the form of $(x - a)(x - b) = 0$, where $a < b$,
(a) if $(x - a)(x - b) > 0$, then $x < a$ or $x > b$,
(b) if $(x - a)(x - b) < 0$, then $a < x < b$.

Example 5

Find the range of values of x for the quadratic inequality $(2x - 1)(x + 4) \geqslant x + 4$ by using
(a) graph sketching method
(b) number line method
(c) table method

Solution

(a) $(2x - 1)(x + 4) \geqslant x + 4$
$2x^2 + 7x - 4 \geqslant x + 4$
$2x^2 + 6x - 8 \geqslant 0$
$x^2 + 3x - 4 \geqslant 0$
$(x + 4)(x - 1) \geqslant 0$

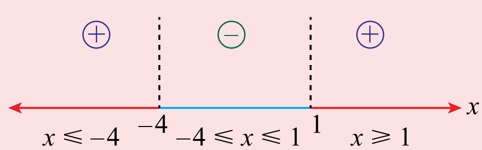


When $(x + 4)(x - 1) = 0$, $x = -4$ or $x = 1$.

The graph will intersect the x-axis at point $x = -4$ and $x = 1$.

Since $(x + 4)(x - 1) \geqslant 0$, thus the range of values of x is determined on the graph above the x-axis. Hence, the range of values of x is $x \leqslant -4$ or $x \geqslant 1$.

(b) Test point -5: $(-5 + 4)(-5 - 1) \geqslant 0$ Test point 0: $(0 + 4)(0 - 1) \leqslant 0$ Test point 2: $(2 + 4)(2 - 1) \geqslant 0$



Since $(x + 4)(x - 1) \geqslant 0$, then the range of values of x is determined on the positive part of the number line.

Hence, the range of values of x is $x \leqslant -4$ or $x \geqslant 1$.

(c)

	Range of values of x		
	$x \leqslant -4$	$-4 \leqslant x \leqslant 1$	$x \geqslant 1$
$(x + 4)$	–	+	+
$(x - 1)$	–	–	+
$(x + 4)(x - 1)$	+	–	+

Since $(x + 4)(x - 1) \geqslant 0$, then the range of values of x is determined on the positive part of the table.

Hence, the range of values of x is $x \leqslant -4$ or $x \geqslant 1$.

Self Practice 2.3

1. Solve each of the following quadratic inequalities by using graph sketching method, number line method or table method.
(a) $x^2 < 4$ (b) $(2 - x)(8 - x) < 0$ (c) $x^2 \leqslant 4x + 12$
(d) $x(x - 2) \geqslant 3$ (e) $(x + 2)^2 < 2x + 7$ (f) $(3x + 1)(5 - x) > 13$

2. Find the range of values of x for $3x^2 - 5x \geqslant 16 + x(2x + 1)$.

Intensive Practice 2.1

Scan the QR code or visit bit.ly/2pSSNs6 for the quiz

1. Solve the quadratic equation $3x(x - 5) = 2x - 1$. Give the answer in three decimal places.

2. Given a quadratic equation $2(x - 5)^2 = 4(x + 7)$,
(a) express the equation in general form, that is $ax^2 + bx + c = 0$.
(b) state the sum of roots and the product of roots of the equation.

3. If α and β are the roots of the quadratic equation $2x^2 + 6x - 7 = 0$, form equations with the following roots:
(a) $\dfrac{1}{2\alpha + 1}, \dfrac{1}{2\beta + 1}$ (b) $\dfrac{5\alpha}{\beta}, \dfrac{5\beta}{\alpha}$ (c) $\alpha + 3\beta, 3\alpha + \beta$

4. If one of the roots of the equation $3x^2 + 19x + k = 0$ is -7, find the value of the constant k.

5. Given the quadratic equation $rx^2 + (r - 1)x + 2r + 3 = 0$, where r is a non-zero integer, find the value of r such that
(a) one root is negative of the other root,
(b) one root is the reciprocal of the other root,
(c) one root is twice the other root.

6. One root of the equation $x^2 - 8x + m = 0$ is three times the other root, find the value of the constant m and the roots.

7. The equation $x^2 + 2x = k(x - 1)$ has non-zero roots where the difference between the roots is 2, find the value of each root and the value of k.

8. The roots of the equation $x^2 + px + 27 = 0$ are in the ratio of $1 : 3$. Find the values of p.

9. Given 3 and $h + 1$ are the roots of the equation $x^2 + (k - 1)x + 9 = 0$, find the possible values of h and k.

10. The two roots of the equation $x^2 - 8x + c = 0$ are α and $\alpha + 3d$. Express c in terms of d.

11. Solve each of the following quadratic inequalities:
(a) $2x^2 \geqslant x + 1$ (b) $(x - 3)^2 \leqslant 5 - x$ (c) $(1 - x)^2 + 2x < 17$

12. Find the value of m and of n for each of the following quadratic inequalities:
(a) $x^2 + mx < n$ which is satisfied by $-3 < x < 4$.
(b) $2x^2 + m > nx$ which is satisfied by $x < -2$ or $x > 5$.

13. Given $y = 2x^2 + bx + 12$ and $y < 0$, if $2 < x < a$, find the value of a and of b.

2.2 Types of Roots of Quadratic Equations

Types of roots of quadratic equations and value of discriminant

You have learned that the roots of a quadratic equation can be found by using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Are the roots of a quadratic equation associated closely to the value of $b^2 - 4ac$ in the formula? Let's explore.

INQUIRY 3 In groups 21st Century Learning

Aim: To explore the relation between types of roots of the quadratic equation $ax^2 + bx + c = 0$ and the value of $b^2 - 4ac$

bit.ly/2RS5Jff

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click one by one on the boxes which display the quadratic equations $y = x^2 + 5x + 4$, $y = x^2 - 6x + 9$ and $y = 9x^2 - 6x + 2$ to display each of the graphs.
3. Observe the positions of those graphs.
4. Identify the values of a, b and c as well as the roots of each graph when $y = 0$.
5. Discuss with the group members on the relation between the values of $b^2 - 4ac$ and the types of roots obtained.
6. Present the findings of your group in front of the class.

From the results of Inquiry 3, note that the types of roots of quadratic equations can be determined from the values of $b^2 - 4ac$ which is known as the **discriminant** and usually denoted by D.

In general:

1. If the discriminant $b^2 - 4ac > 0$, the equation has two different real roots.
2. If the discriminant $b^2 - 4ac = 0$, the equation has two equal real roots.
3. If the discriminant $b^2 - 4ac < 0$, the equation has no real roots.

For the quadratic equation $9x^2 - 6x + 2 = 0$ which has no roots, note that the value of the discriminant is negative. Since $\sqrt{-36}$ is not a real number, then this quadratic equation has no real roots. The square root of a negative number is known as an imaginary root and it is represented by $i = \sqrt{-1}$. Then, the roots of the quadratic equation $9x^2 - 6x + 2$ can be written as

$$x = \frac{6 \pm \sqrt{-36(-1)}}{18} = \frac{6 \pm 6i}{18} = \frac{1 \pm i}{3}.$$

Smart TIPS

When the discriminant $b^2 - 4ac \geqslant 0$, the equation has real roots.

Mind Challenge

What is the type of roots of a quadratic equation if the discriminant $b^2 - 4ac \leqslant 0$?

BRAINSTORMING

Determine the roots of the following quadratic equations. Give your answers in terms of imaginary number, i, $i = \sqrt{-1}$.

(a) $x^2 + 4x + 5 = 0$
(b) $x^2 - 2x + 3 = 0$
(c) $2x^2 - 6x + 5 = 0$

Example 6

Determine the type of roots for each of the following quadratic equation.

(a) $x^2 + 5x - 6 = 0$
(b) $-4x^2 + 4x - 1 = 0$
(c) $2x^2 - 4x + 5 = 0$

Solution

(a) $x^2 + 5x - 6 = 0$ with $a = 1$, $b = 5$ and $c = -6$

$b^2 - 4ac = 5^2 - 4(1)(-6)$
$= 49\ (> 0)$

Thus, the equation $x^2 + 5x - 6 = 0$ has two real and different roots.

(b) $-4x^2 + 4x - 1 = 0$ with $a = -4$, $b = 4$ and $c = -1$

$b^2 - 4ac = 4^2 - 4(-4)(-1)$
$= 0$

Thus, the equation $-4x^2 + 4x - 1 = 0$ has two equal real roots.

(c) $2x^2 - 4x + 5 = 0$ with $a = 2$, $b = -4$ and $c = 5$

$b^2 - 4ac = (-4)^2 - 4(2)(5)$
$= -24\ (< 0)$

Thus, the equation $2x^2 - 4x + 5 = 0$ has no real roots.

Mind Challenge

Why must the value of the discriminant need to be found first when determining the types of roots of a quadratic equation?

Tech Whizz

Check your answers by *Mathpapa* application which can be downloaded from your mobile phone.

bit.ly/2LGClgg

Self Practice 2.4

1. Find the discriminant and determine the types of roots for each of the following quadratic equation:

(a) $x^2 + 4x + 1 = 0$
(b) $x^2 = 8(x - 2)$
(c) $5x^2 + 4x + 6 = 0$
(d) $-3x^2 + 7x + 5 = 0$
(e) $-x^2 + 10x - 25 = 0$
(f) $(2x - 1)(x + 3) = 0$

Solving problems involving types of roots of quadratic equations

The discriminant, D which determines the types of roots of a quadratic equation $ax^2 + bx + c = 0$ can be used to:

(a) Find the value of a variable in the quadratic equation.
(b) Derive a relation.

Example 7

(a) The quadratic equation $x^2 + k + 3 = kx$, where k is a constant, has two equal real roots. Find the possible values of k.
(b) The roots of the equation $(p + 2)x^2 - 2px = 3 - p$, where p is a constant, are real and different. Find the range of values of p.
(c) Given the quadratic equation $x^2 + 4x + 13 = m(2 - x)$, where m is a constant, has no real roots. Find the range of values of m.

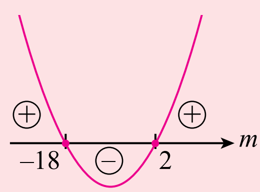
Solution

(a)
$$x^2 + k + 3 = kx$$
$$x^2 - kx + k + 3 = 0$$ ← Arrange the equation in the general form
$$a = 1,\ b = -k \text{ and } c = k + 3$$
$$b^2 - 4ac = 0$$ ← Two real and equal roots
$$(-k)^2 - 4(1)(k + 3) = 0$$
$$k^2 - 4k - 12 = 0$$
$$(k + 2)(k - 6) = 0$$
$$k = -2 \quad \text{or} \quad k = 6$$

(b)
$$(p + 2)x^2 - 2px = 3 - p$$
$$(p + 2)x^2 - 2px + p - 3 = 0$$ ← Arrange the equation in the general form
$$a = p + 2,\ b = -2p \text{ and } c = p - 3$$
$$b^2 - 4ac > 0$$ ← Two real and different roots
$$(-2p)^2 - 4(p + 2)(p - 3) > 0$$
$$4p^2 - 4(p^2 - p - 6) > 0$$
$$4p + 24 > 0$$
$$p > -6$$

(c)
$$x^2 + 4x + 13 = m(2 - x)$$
$$x^2 + 4x + 13 = 2m - mx$$
$$x^2 + 4x + mx + 13 - 2m = 0$$
$$x^2 + (4 + m)x + 13 - 2m = 0$$ ← Arrange the equation in the general form
$$a = 1,\ b = 4 + m \text{ and } c = 13 - 2m$$
$$b^2 - 4ac < 0$$ ← No real roots
$$(4 + m)^2 - 4(1)(13 - 2m) < 0$$
$$16 + 8m + m^2 - 52 + 8m < 0$$
$$m^2 + 16m - 36 < 0$$
$$(m + 18)(m - 2) < 0$$

Thus, the range of values of m is $-18 < m < 2$.



Example 8

Given the equation $x^2 - 4ax + 5b = 0$ has two real and equal roots, express a in terms of b.

Solution

$x^2 - 4ax + 5b = 0$ where $a = 1$, $b = -4a$ and $c = 5b$.

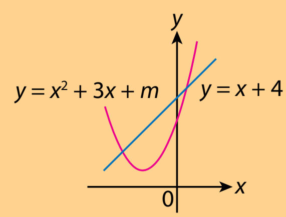
Since the equation has two real and equal roots,
$$b^2 - 4ac = 0$$
$$(-4a)^2 - 4(1)(5b) = 0$$
$$16a^2 - 20b = 0$$
$$16a^2 = 20b$$
$$a^2 = \frac{5}{4}b$$
$$a = \pm\frac{1}{2}\sqrt{5b}$$

BRAINSTORMING

By assuming $b^2 - 4ac \geqslant 0$, show that the solutions of the equation $ax^2 + bx + c = 0$ are the reciprocal of the solutions of the equation $cx^2 + bx + a = 0$.

MATHEMATICS POCKET

Consider a line $y = x + 4$ that crossed a curve $y = x^2 + 3x + m$ as shown in the diagram below.



To find the range of values of m, solve both equations simultaneously.
$$x^2 + 3x + m = x + 4$$
$$x^2 + 2x + m - 4 = 0$$
This quadratic equation has two real and different roots.
Thus,
$$b^2 - 4ac > 0$$
$$2^2 - 4(1)(m - 4) > 0$$
$$4 - 4m + 16 > 0$$
$$4m < 20$$
$$m < 5$$
Hence, the range of values of m is $m < 5$. Discuss with your friends and find the values of m or the range of values of m for the following cases:
- Line $y = mx - 5$ touches a point on the curve $2y = x^2 - 1$.
- Line $y = mx + 4$ crosses a curve $5x^2 - xy = 2$ at two points.
- Line $y = 2x + 3$ does not cross the curve $x^2 + xy = m$.

2.2.2

Self Practice 2.5

1. Find the values or range of values of p such that the equation
(a) $9x^2 + p + 1 = 4px$ has two equal roots,
(b) $x^2 + (2x + 3)x = p$ has two real and different roots,
(c) $x^2 + 2px + (p - 1)(p - 3) = 0$ has no real roots.

2. Find the range of values of k if the equation $x^2 + k = kx - 3$ has two real and different roots. State the values of k if the equation has two real and equal roots.

3. The quadratic equation $x^2 + hx + k = 0$ has roots of -2 and 6, find
(a) the value of h and of k,
(b) the range of values of c such that the equation $x^2 + hx + k = c$ has no real roots.

4. The equation $hx^2 + 3hx + h + k = 0$, where $h \neq 0$, has two real and equal roots. Express k in terms of h.

5. Given the quadratic equation $ax^2 - 5bx + 4a = 0$, where a and b are constants, has two real and equal roots, find $a : b$.

Intensive Practice 2.2

Scan the QR code or visit bit.ly/2nClGqQ for the quiz

1. Determine the types of roots for the following quadratic equations.
(a) $x^2 - 8x + 16 = 0$ (b) $(x - 2)^2 = 3$ (c) $2x^2 + x + 4 = 0$

2. The following quadratic equations have two real and equal roots. Find the values of k.
(a) $x^2 + kx = 2x - 9$ (b) $kx^2 + (2k + 1)x + k - 1 = 0$

3. The following quadratic equations have two real and different roots. Find the range of values of r.
(a) $x(x + 1) = rx - 4$ (b) $x^2 + x = 2rx - r^2$

4. Find the range of values of p if the following equations have no real roots.
(a) $(1 - p)x^2 + 5 = 2x$ (b) $4px^2 + (4p + 1)x + p - 1 = 0$

5. The equation $kx^2 - 10x + 6k = 5$, where k is a constant, has two real and equal roots.
(a) Find the values of k.
(b) Hence, find the roots of the equation by using the smallest value of k obtained in (a).

6. The quadratic equation $x(x - 4) + 2n = m$ where m and n are constants, has two real and equal roots. Express m in terms of n.

7. The quadratic equation $x^2 + bx + c = 0$ where b and c are positive integers, has a discriminant of 16 and $b - c = -4$. Find
(a) the possible values of b and c,
(b) the corresponding roots of the equations.

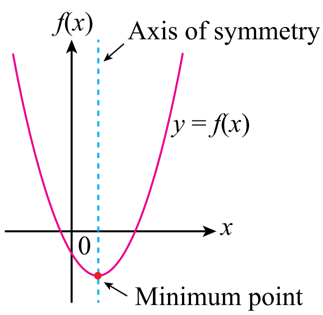
8. The quadratic equation $2x^2 - 5x + c = 0$ where c is a positive integer, has no real roots.
(a) Find two possible values of c, that is c_1 and c_2.
(b) Based on the values of c_1 and c_2 in (a), does the equation $2x^2 - 5x + \frac{1}{2}(c_1 + c_2) = 0$ have two real roots? Explain.

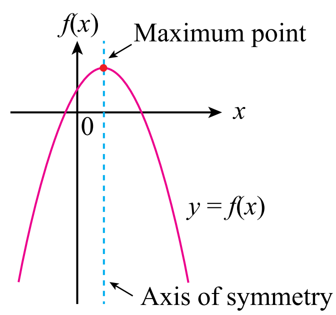
2.3 Quadratic Functions

A ball is thrown into the net. What can you observe from the path of the ball? If you observe the path of the ball, it follows the shape of a parabola. The path or the curve is the shape of the graph of a quadratic function. What are other examples that involve the shape of a parabola?

Analysing the effect of changes of *a*, *b* and *c* towards the shape and position of the graph for $f(x) = ax^2 + bx + c$

The general form of a quadratic function is a function in the form of $f(x) = ax^2 + bx + c$ where a, b, and c are constants and $a \neq 0$. The shape of the graph of a quadratic function is a parabola which is symmetrical about the axis that passes through the minimum point or maximum point.





MATHEMATICS POCKET

The highest power of a quadratic function is the same as the highest power of a quadratic equation, that is 2.

What are the effects on the shape and position of the quadratic function graph if the values of a, b and c change? Let's explore.

INQUIRY 4 In groups | 21st Century Learning

Aim: To explore the effects of changes in the values of a, b and c towards the shape and position of the quadratic function graph

Instructions:

1. Scan the QR code or visit the link on the right.
2. Observe the graph of function $f(x) = ax^2 + bx + c$, where $a = 1$, $b = 2$ and $c = 3$.
3. In groups, analyse the changes of shape and position of the quadratic function graph based on the following instructions:
   (a) Drag slide a to the left and to the right without changing slide b and slide c.
   (b) Drag slide b to the left and to the right without changing slide a and slide c.
   (c) Drag slide c to the left and to the right without changing slide a and slide b.
4. Make a generalisation on the effects of changes in the values of a, b and c on the shape and position of the graph of $f(x) = ax^2 + bx + c$.
5. Present the findings of your group in front of the class and discuss with other groups.

bit.ly/324HT0w

2.3.1

From the results of Inquiry 4, the following findings are obtained.

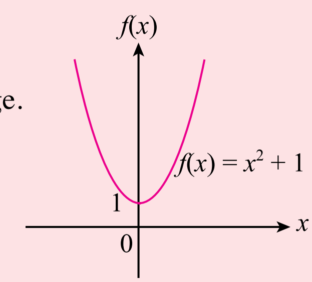
	Changes in shape and position of the graph of function $f(x) = ax^2 + bx + c$
Only the value of a changes	• Change in value of a affects the shape and width of the graph, however the y-intercept remains unchanged. • When $a > 0$, the shape of the graph is $\cup$ which passes through the minimum point and when $a < 0$, the shape of the graph is $\cap$ which passes through the maximum point. • For the graph $a > 0$, for example $a = 1$, when the value of a is larger than 1, the width of the graph decreases. Conversely, when the value of a is smaller than 1 and approaches 0, the width of the graph increases. • For the graph $a < 0$, for example $a = -1$, when the value of a is smaller than -1, the width of the graph decreases. Conversely, when the value of a increases from -1 and approaches 0, the width of the graph increases.
Only the value of b changes	• Change in value of b only affects the position of vertex with respect to the y-axis, however the shape of the graph and the y-intercept are unchanged. • When $b = 0$, the vertex is on the y-axis. • For the graph $a > 0$, when $b > 0$, the vertex is on the left side of the y-axis and when $b < 0$, the vertex is on the right side of the y-axis. • For the graph $a < 0$, when $b > 0$, the vertex is on the right side of the y-axis and when $b < 0$, the vertex is on the left side of the y-axis.
Only the value of c changes	• Change in value of c only affects the position of graph either vertically upwards or vertically downwards. • The shape of the graph is unchanged.

Example 9

The diagram shows the sketch of the graph for $f(x) = x^2 + 1$ where $a = 1$, $b = 0$ and $c = 1$. Make an analysis and a generalisation on the shape and position of the graph when the following values change. Hence, sketch the graph.

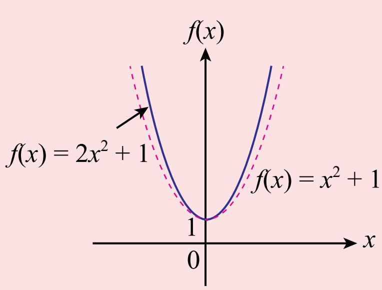
(a) The value of a becomes
  (i) 2, (ii) $\frac{1}{2}$.

(b) The value of c becomes 3.

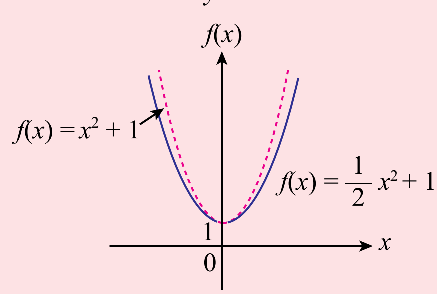


Solution

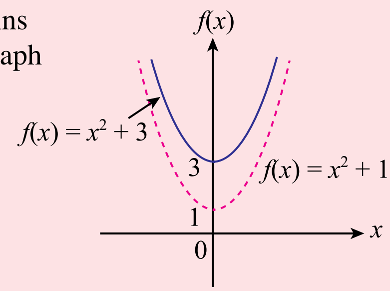
(a) (i) When a changes from 1 to 2, the width of the graph decreases. The y-intercept does not change and the vertex is on the y-axis.



(ii) When a changes from 1 to $\frac{1}{2}$, the width of graph increases. The y-intercept does not change and the vertex is on the y-axis.

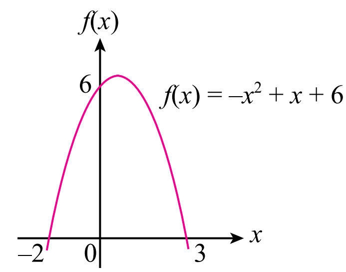


(b) When c changes from 1 to 3, the shape of the graph remains unchanged. The only change is the position, that is, the graph moves 2 units upwards.



Self Practice 2.6

1. The diagram on the right shows the graph for $f(x) = -x^2 + x + 6$, where $a = -1$, $b = 1$ and $c = 6$. Sketch the graph of $f(x)$ formed when the following values change.



(a) The value of a changes to

(i) -3 (ii) $-\frac{1}{4}$,

(b) The value of b changes to -1,

(c) The value of c changes to -2.

Make a generalisation from the changes in the shape and position of the graphs obtained.

Relating the position of the graph of a quadratic function and the types of roots

You have learned that the discriminant $b^2 - 4ac$ of a quadratic equation $ax^2 + bx + c = 0$ can be used to determine the types of roots. Let's see how the types of roots of a quadratic equation can determine the position of the graph of a quadratic function $f(x) = ax^2 + bx + c$ with respect to the x-axis.

INQUIRY 5 In groups

Aim: To explore the relation between the position of the graph of quadratic function and types of roots

Instructions:

1. Each group has to choose only one case out of the following two cases.

Case 1	Case 2
(a) $f(x) = x^2 + 4x + 4$	(a) $f(x) = -x^2 + 2x - 1$
(b) $f(x) = 2x^2 + 7x - 4$	(b) $f(x) = -2x^2 - 8x - 5$
(c) $f(x) = x^2 - 6x + 12$	(c) $f(x) = -x^2 + 6x - 10$

2. By using a dynamic geometry software, plot the graph of each quadratic function in the case chosen.

3. Observe the shapes of the graphs obtained and the respective roots.

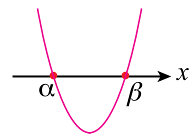
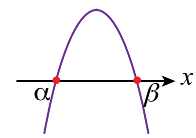
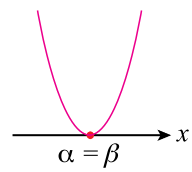
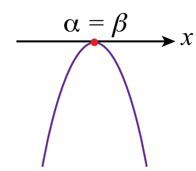
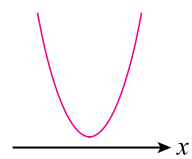
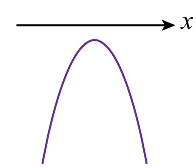
4. State the relation between the value of $b^2 - 4ac$, types of roots and the number of intersection points on the x-axis.

5. From the relation, state the position of the graph of the quadratic function obtained.

6. Compare the findings of your group with other groups of different case and make a conclusion on the comparison made.

2.3.1 2.3.2

From the results of Inquiry 5, the relation between the position of graph of quadratic function $f(x) = ax^2 + bx + c$ on the x-axis and the types of roots can be summarised as shown in the table below.

Discriminant, $b^2 - 4ac$	**Types of roots and position of graph**	**Position of graph of function $f(x) = ax^2 + bx + c$**	
		$a > 0$	**$a < 0$**
$b^2 - 4ac > 0$	• Two real and different roots • The graph intersects the x-axis at two different points.		
$b^2 - 4ac = 0$	• Two real and equal roots • The graph touches the x-axis at one point only.		
$b^2 - 4ac < 0$	• No real roots • The graph does not intersect at any point on the x-axis.		

Example 10

Determine the types of roots for each of the following quadratic functions when $f(x) = 0$. Then, sketch the graph and make a generalisation on the position of graph at the x-axis.

(a) $f(x) = 2x^2 + x - 5$ (b) $f(x) = -x^2 + 2x - 1$

Solution

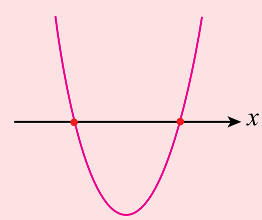
(a) $f(x) = 2x^2 + x - 5$

$a = 2, b = 1, c = -5$

$b^2 - 4ac = (1)^2 - 4(2)(-5)$

$= 41\ (> 0)$

The quadratic function has two real and different roots. Since $a > 0$, thus the graph of $f(x)$ is a parabola which passes through the minimum point and intersects the x-axis at two points.



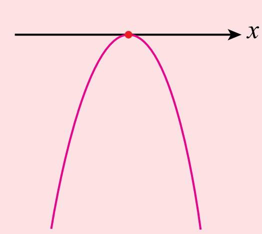
(b) $f(x) = -x^2 + 2x - 1$

$a = -1, b = 2, c = -1$

$b^2 - 4ac = (2)^2 - 4(-1)(-1)$

$= 0$

The quadratic function has two real and equal roots. Since $a < 0$, thus the graph of $f(x)$ is a parabola which passes through the maximum point and intersects the x-axis at one point.



Example 11

(a) Find the values of m such that the x-axis is the tangent to the graph of a quadratic function $f(x) = (m + 1)x^2 + 4(m - 2)x + 2m$.

(b) Find the range of values of k if the graph of a quadratic function $f(x) = 2x^2 + 5x + 3 - k$ has no x-intercept.

(c) Find the range of values of p if the graph of a quadratic function $f(x) = x^2 + px + p + 3$ has two x-intercepts.

Solution

(a) The graph of a quadratic function $f(x) = (m + 1)x^2 + 4(m - 2)x + 2m$ such that the x-axis is a tangent to the graph which means the function has two real and equal roots.

For two real and equal roots:

$$\begin{aligned} b^2 - 4ac &= 0 \\ (4m - 8)^2 - 4(m + 1)(2m) &= 0 \\ 16m^2 - 64m + 64 - 8m^2 - 8m &= 0 \\ 8m^2 - 72m + 64 &= 0 \\ m^2 - 9m + 8 &= 0 \\ (m - 1)(m - 8) &= 0 \\ m = 1 \text{ or } m &= 8 \end{aligned}$$

(b) The graph of a quadratic function $f(x) = 2x^2 + 5x + 3 - k$ has no x-intercept, which means the function has no real roots.

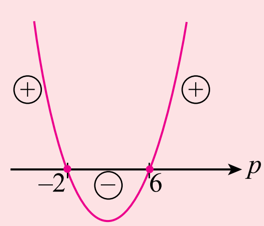
For no real roots:

$$\begin{aligned} b^2 - 4ac &< 0 \\ 5^2 - 4(2)(3 - k) &< 0 \\ 25 - 24 + 8k &< 0 \\ 1 + 8k &< 0 \\ 8k &< -1 \\ k &< -\frac{1}{8} \end{aligned}$$

(c) The graph of a quadratic function $f(x) = x^2 + px + p + 3$ has two x-intercepts which means the function has two real and different roots.

For two real and different roots:

$$\begin{aligned} b^2 - 4ac &> 0 \\ p^2 - 4(1)(p + 3) &> 0 \\ p^2 - 4p - 12 &> 0 \\ (p + 2)(p - 6) &> 0 \\ p < -2 \quad \text{or} \quad p &> 6 \end{aligned}$$



BRAINSTORMING

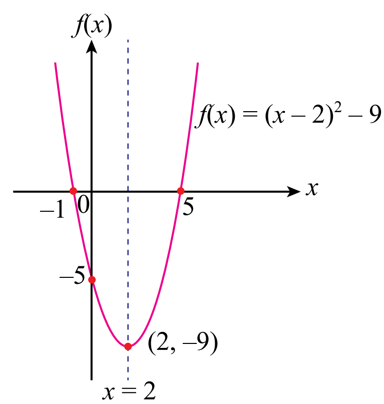
What are the rules of a quadratic function $f(x) = ax^2 + bx + c$ so that it is always positive or always negative for all real values of x? Discuss.

2.3.2

Self Practice 2.7

1. Determine the types of roots for each of the following quadratic function. Sketch the graph and make generalisation on the position of the graph on the x-axis.
   (a) $f(x) = -3x^2 + 6x - 3$ (b) $f(x) = x^2 + 2x - 3$ (c) $f(x) = 4x^2 - 8x + 5$
2. Find the possible values of h if the graphs of the following quadratic functions touch the x-axis at only one point.
   (a) $f(x) = x^2 - 2hx + 2 + h$ (b) $f(x) = x^2 - (h + 3)x + 3h + 1$
3. Find the range of values of q if the graph of the following quadratic functions intersect the x-axis at two points.
   (a) $f(x) = 5x^2 - (qx + 4)x - 2$ (b) $f(x) = (q + 2)x^2 + q(1 - 2x) - 5$
4. Find the range of values of r if the graphs of the following quadratic functions do not intersect the x-axis.
   (a) $f(x) = rx^2 + 4x - 6$ (b) $f(x) = rx^2 + (2r + 4)x + r + 7$

Making relation between the vertex form of a quadratic function $f(x) = a(x - h)^2 + k$ with the other forms of quadratic functions



The diagram on the right shows the sketch of graph of a quadratic function in the vertex form, $f(x) = (x - 2)^2 - 9$. Since $a > 0$, the graph of the quadratic function is in the shape of $\cup$. Note that the graph of this quadratic function has its vertex at the minimum point $(2, -9)$ and the equation of the axis of symmetry, $x = 2$.

The **vertex form** is a quadratic function in the form of $f(x) = a(x - h)^2 + k$, where a, h and k are constants. The vertex is (h, k) and it is symmetrical about the line $x = h$.

When $a > 0$, the vertex (h, k) is the minimum point and k is the minimum value of $f(x)$. When $a < 0$, the vertex (h, k) is the maximum point and k is the maximum value of $f(x)$.

Other than the vertex form, the quadratic function can be written in the following form:

Form of quadratic functions

- **General form**, $f(x) = ax^2 + bx + c$, where a, b and c are constants with a vertex at the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ and symmetrical about the line $x = -\frac{b}{2a}$.

- **Intercept form**, $f(x) = a(x - p)(x - q)$, where a, p and q are constants. p and q are the roots or x-intercepts for $f(x)$, its vertex is at the point $\left(\frac{p + q}{2}, f\left(\frac{p + q}{2}\right)\right)$ and symmetrical about the line $x = \frac{p + q}{2}$.

What is the relation that exists between vertex form of quadratic functions with the general form and the intercept form? Let's explore.

INQUIRY 6 In groups

Aim: To explore the relation between the vertex form of a quadratic function with the general form and intercept form

Instructions:

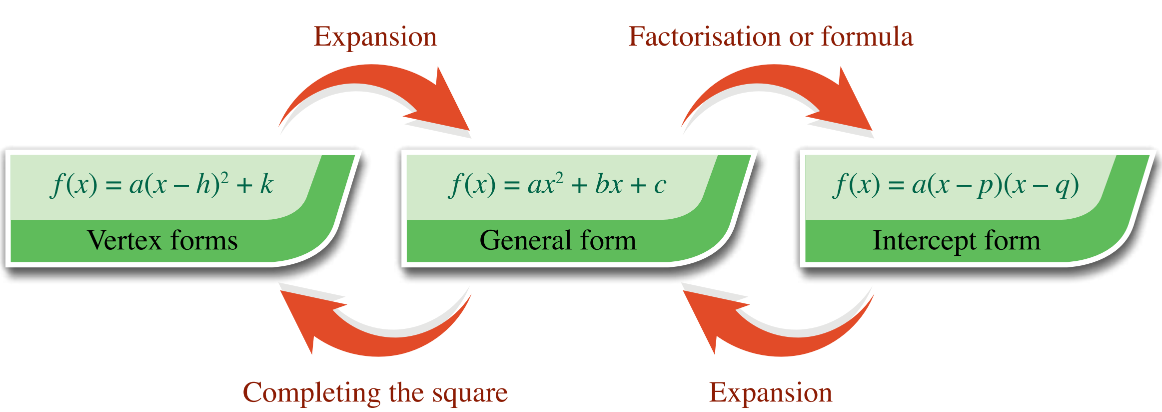
1. Consider a quadratic function in the vertex form, $f(x) = (x - 4)^2 - 4$.
2. In each group, discuss and express the quadratic function in the vertex form into general form and intercept form.
3. Then, copy and complete the table below.

Shape of quadratic function	Quadratic function	*x*-intercept	*y*-intercept	Vertex	Axis of symmetry
Vertex form	$f(x) = (x - 4)^2 - 4$				
General form					
Intercept form					

4. Sketch the graphs for each of the forms of the quadratic function. Check the sketching of graphs by using dynamic geometry software.
5. Compare the graphs of quadratic function which are plotted in the vertex form, general form and intercept form.
6. Carry out a brainstorming session in the group and obtain a conclusion on the relation that exists between quadratic function in the vertex form with the general form and intercept form.

From the results of Inquiry 6, it is found that the quadratic function $f(x) = (x - 4)^2 - 4$ in vertex form, general form and intercept form produced the same graph when they are sketched.

In expressing quadratic function in the vertex form to general form and intercept form or vice versa, the following methods can be used:



Example 12

Express quadratic function, $f(x) = 2\left(x + \frac{9}{4}\right)^2 - \frac{1}{8}$ in the intercept form, $f(x) = a(x - p)(x - q)$, where a, p and q are constants and $p < q$. Hence, state the values of a, p and q.

Solution

Convert the vertex form of the quadratic function into the general form first.

$$\begin{aligned} f(x) &= 2\left(x + \frac{9}{4}\right)^2 - \frac{1}{8} \\ &= 2\left(x^2 + \frac{9}{2}x + \frac{81}{16}\right) - \frac{1}{8} \\ &= 2x^2 + 9x + 10 \quad \leftarrow \text{General form} \\ &= (2x + 5)(x + 2) \\ &= 2\left(x + \frac{5}{2}\right)(x + 2) \quad \leftarrow \text{Intercept form} \end{aligned}$$

Thus, the quadratic function in the intercept form for $f(x) = 2\left(x + \frac{9}{4}\right)^2 - \frac{1}{8}$ can be expressed as $f(x) = 2\left(x + \frac{5}{2}\right)(x + 2)$, where $a = 2$, $p = -\frac{5}{2}$ and $q = -2$.

Mind Challenge

Not all the vertex forms or general forms can be expressed in intercept form, only graphs which have x-intercept can be expressed. Do you agree with the statement?

Alternative Method

$$\begin{aligned} f(x) &= 2\left(x + \frac{9}{4}\right)^2 - \frac{1}{8} \\ &= 2\left[\left(x + \frac{9}{4}\right)^2 - \frac{1}{4^2}\right] \end{aligned}$$

Use $a^2 - b^2 = (a + b)(a - b)$

$$\begin{aligned} f(x) &= 2\left(x + \frac{9}{4} + \frac{1}{4}\right)\left(x + \frac{9}{4} - \frac{1}{4}\right) \\ &= 2\left(x + \frac{10}{4}\right)\left(x + \frac{8}{4}\right) \\ &= 2\left(x + \frac{5}{2}\right)(x + 2) \end{aligned}$$

Example 13

Express $f(x) = -3x^2 + 2x + 1$ as $f(x) = a(x - h)^2 + k$, where a, h and k are constants. Hence, determine the values of a, h and k.

Solution

$f(x) = -3x^2 + 2x + 1$

Make sure the coefficient of x^2 is 1 before completing the square.

$$\begin{aligned} f(x) &= -3x^2 + 2x + 1 \\ &= -3\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right) \quad \leftarrow \text{Factorise } -3 \text{ from } -3x^2 + 2x + 1 \\ &= -3\left[x^2 - \frac{2}{3}x + \left(\frac{-1}{3}\right)^2 - \left(\frac{-1}{3}\right)^2 - \frac{1}{3}\right] \quad \leftarrow \text{Add and subtract } \left(\frac{\text{coefficient of } x}{2}\right)^2 \\ &= -3\left[\left(x - \frac{1}{3}\right)^2 - \left(\frac{-1}{3}\right)^2 - \frac{1}{3}\right] \\ &= -3\left[\left(x - \frac{1}{3}\right)^2 - \frac{4}{9}\right] \\ &= -3\left(x - \frac{1}{3}\right)^2 + \frac{4}{3} \end{aligned}$$

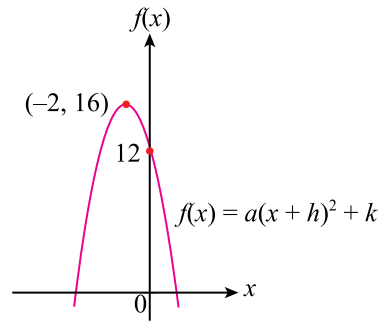
Thus, $a = -3$, $h = \frac{1}{3}$ and $k = \frac{4}{3}$.

BRAINSTORMING

By using completing the square method, show that the equation of the axis of symmetry for $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

Self Practice 2.8

1. Given $f(x) = 2(x - 3)^2 - 8 = a(x - p)(x - q)$ for all values of x, find the values of the constants a, p and q where $p < q$.
2. Express each of the following vertex form into general form and intercept form.
   (a) $f(x) = (x - 2)^2 - 1$ (b) $f(x) = 9 - (2x - 1)^2$ (c) $f(x) = 2(x + 1)^2 - 18$
3. Find the vertex of the function $f(x) = -\frac{1}{2}(x + 4)^2 - 5$ and convert it into general form.
4. The diagram on the right shows the graph of quadratic function $f(x) = a(x + h)^2 + k$, where a, h and k are constants. Given (–2, 16) is the maximum point of the graph.
   (a) State the values of a, h and k.
   (b) Hence, express the function in general form, $f(x) = ax^2 + bx + c$ and intercept form, $f(x) = a(x - p)(x - q)$.



5. Express each of the following in the vertex form, $f(x) = a(x + h)^2 + k$, where a, h and k are constants.
   (a) $f(x) = x^2 - x - 6$ (b) $f(x) = -x^2 - 2x + 4$ (c) $f(x) = -2x^2 - x + 6$
   (d) $f(x) = 3x^2 - 2x - 9$ (e) $f(x) = (x + 2)(6 - x)$ (f) $f(x) = 2(x + 4)(x - 2)$

Analysing the effect of change of a, h and k on the shape and position of graph for $f(x) = a(x - h)^2 + k$

The quadratic function in the vertex form, $f(x) = a(x + h)^2 + k$, where a, h and k are constants has its vertex at (h, k) and it is symmetrical about the line $x = h$. What will happen to the shape and position of the graph of function $f(x)$ when the values of a, h and k are changed?

INQUIRY 7 | In groups | 21st Century Learning

Aim: To explore the effect of change in the values of a, h and k on the shape and position of the graph $f(x) = a(x - h)^2 + k$

Instructions:

1. Scan the QR code or visit the link on the right.
2. Observe the graph of function $f(x) = a(x - h)^2 + k$ where $a = 2$, $h = 3$ and $k = 1$.
3. In groups, make an analysis and state the observation on the shape and position of the graph based on each of the following instructions:
   (a) Drag slide a to the left and to the right without changing slide h and slide k.
   (b) Drag slide h to the left and to the right without changing slide a and slide k.
   (c) Drag slide k to the left and to the right without changing slide a and slide h.
4. What happens to the axis of symmetry, minimum value or maximum value of the graph when the values of a, h or k change?
5. Make a generalisation on the effect of change in the values of a, h and k on the shape and position of the graph $f(x) = a(x - h)^2 + k$.

bit.ly/2OrShvq

2.3.3 2.3.4

From the results of Inquiry 7, it is found that:

	Changes in shape and position of the graph of function $f(x) = a(x - h)^2 + k$
Only the value of a changes	• Change in value of a affects the shape and width of graph. • When $a > 0$, the graph is in the shape of $\cup$ which passes through the minimum point and when $a < 0$, the graph is in the shape of $\cap$ which passes through the maximum point. • For the graph $a > 0$, for example $a = 2$, when the value of a is increased to more than 2, the width of the graph decreases. Conversely, when the value of a decreases from 2 and approaches 0, the width of the graph increases. • For the graph $a < 0$, for example $a = -2$, when the value of a is decreased to smaller than -2, the width of the graph decreases. Conversely, when the value of a increases from -2 and approaches 0, the width of the graph increases. • The axis of symmetry and the maximum or minimum value remain unchanged.
Only the value of h changes	• The change in the values of h only shows the horizontal movement of the graph. • When the value of h increases, the graph will move to the right whereas when the value of h decreases, the graph will move to the left. • The position of the axis of symmetry changes but the minimum or maximum values remain unchanged.
Only the value of k changes	• The change in the values of k only shows vertical movement of the graph. • When the values of k increases, the graph will move upwards whereas when the value of k decreases, the graph will move downwards. • The maximum and minimum values change but the axis of symmetry remain unchanged.

Example 14

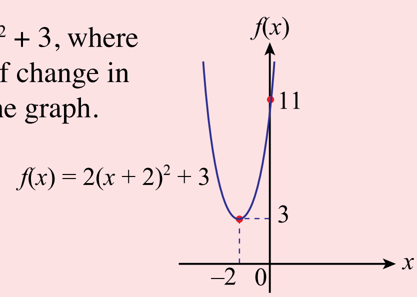
The diagram on the right shows the graph of $f(x) = 2(x + 2)^2 + 3$, where $a = 2$, $h = -2$ and $k = 3$. Make generalisation on the effect of change in each of the following values on the shape and position of the graph.

(a) The value of a changes to

(i) 6, (ii) $\frac{1}{2}$.

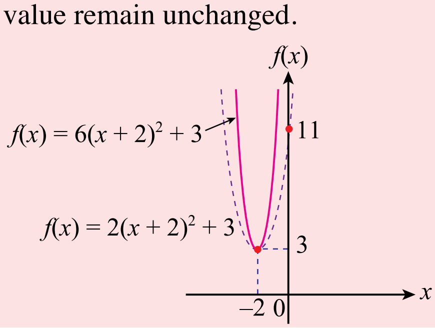
(b) The value of h changes to -6.

(c) The value of k changes to 8.

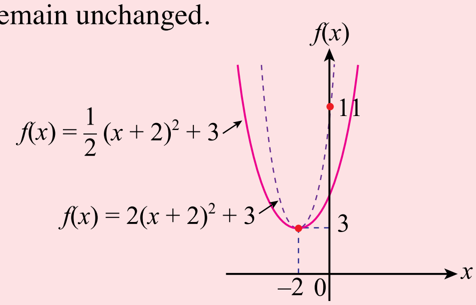


Solution

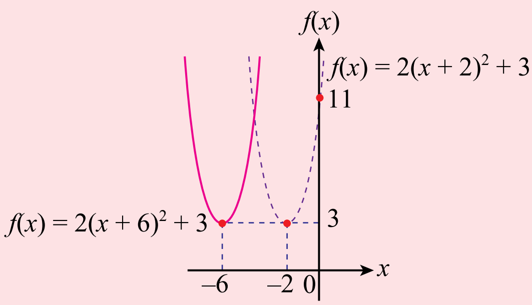
(a) (i) When a changes from 2 to 6, the width of the graph decreases. The axis of symmetry and the minimum value remain unchanged.



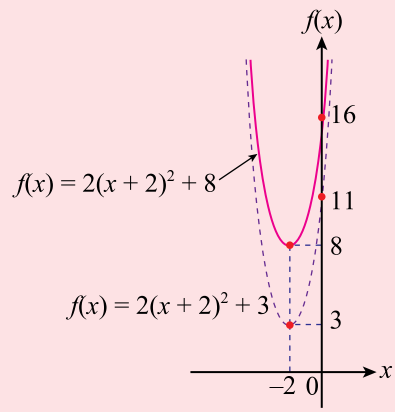
(ii) When a changes from 2 to $\frac{1}{2}$, the width of the graph increases. The axis of symmetry and the minimum value remain unchanged.



(b) When h changes from –2 to –6, the graph with the same shape moves horizontally 4 units to the left. The equation of the axis of symmetry becomes $x = -6$ and the minimum value remains unchanged, that is 3.



(c) When k changes from 3 to 8, the graph with the same shape moves vertically 5 units upwards. The minimum value becomes 8 and the equation of the axis of symmetry is still the same, that is $x = -2$.



Self Practice 2.9

1. The diagram on the right shows the graph of $f(x) = -3(x - 2)^2 + 4$ where $a = -3$, $h = 2$ and $k = 4$.

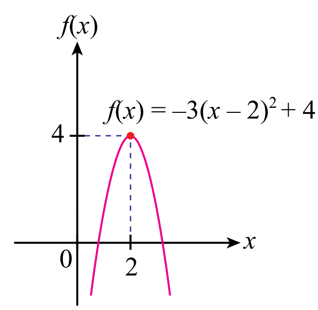
(a) Determine the coordinate of maximum point and the equation of the axis of symmetry.

(b) Make generalisation on the shape and position of the graph when the following values change. Hence, sketch the graphs.

(i) The value of a changes to –10.

(ii) The value of h changes to 5.

(iii) The value of k changes to –2.

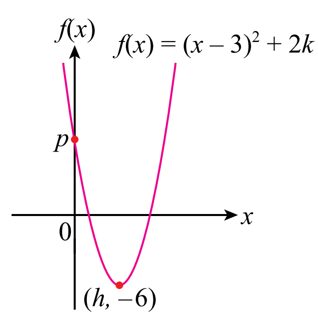


2. The diagram on the right shows the graph of function $f(x) = (x - 3)^2 + 2k$, where k is a constant. Given $(h, -6)$ is the minimum point of the graph.

(a) State the values of h, k and p.

(b) If the graph moves 2 units to the right, determine the equation of the axis of symmetry for the curve.

(c) If the graph moves 5 units upwards, determine the minimum value.



3. Compare the graph of each of the following quadratic functions to the graph of $f(x) = x^2$ with its vertex at $(0, 0)$.

(a) $f(x) = \frac{1}{2}(x - 6)^2$ (b) $f(x) = 3(x - 1)^2 + 5$ (c) $f(x) = \frac{1}{4}(x + 1)^2 - 4$

Sketching the graphs of quadratic functions

The graph of a quadratic function in various shapes can be sketched by following the steps below:

1. Identify the value of a to determine the shape of the graph of a quadratic function.
2. Find the value of discriminant, $b^2 - 4ac$ to determine the position of the graph.
3. Determine the vertex.
4. Determine the intersection point on the x-axis by solving the equation of quadratic function $f(x) = 0$.
5. Find the value of $f(0)$ to determine the y-intercept.
6. Plot the points obtained on the Cartesian plane and draw a smooth parabola that is symmetrical at the vertical line passing through the vertex.

Example 15

Sketch the graph of quadratic function $f(x) = -x^2 + 4x + 12$.

Solution

$a < 0$, so $f(x)$ has a maximum point

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(-1)(12) \\ &= 16 + 48 \\ &= 64\ (> 0) \end{aligned}$$

The curve intersects the x-axis at two different points.

$$\begin{aligned} f(x) &= -x^2 + 4x + 12 \\ &= -(x^2 - 4x - 12) \\ &= -\left[x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 - 12\right] \\ &= -(x - 2)^2 + 16 \end{aligned}$$

Maximum point is (2, 16) and the equation of the axis of symmetry, $x = 2$.

$$\begin{aligned} f(x) &= 0 \\ -x^2 + 4x + 12 &= 0 \\ (-x + 6)(x + 2) &= 0 \end{aligned}$$

$-x + 6 = 0$ or $x + 2 = 0$

$x = 6$ $\quad$ $x = -2$

The intersection on the x-axis are $x = -2$ and $x = 6$.

$$\begin{aligned} f(0) &= -(0)^2 + 4(0) + 12 \\ &= 12 \end{aligned}$$

The graph intersects the y-axis at (0, 12).

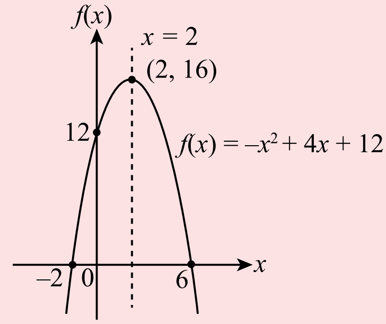
BRAINSTORMING

How to sketch a quadratic function graph in Example 15 in the domain $-3 \leqslant x \leqslant 7$?

BRAINSTORMING

Without expressing to vertex form, how to find the vertex of a quadratic function in general form, $f(x) = ax^2 + bx + c$ and intercept form, $f(x) = a(x - p)(x - q)$? Discuss.

The curve is sketched as shown in the diagram on the right.



Self Practice 2.10

1. Sketch the graphs of the following quadratic functions.

(a) $f(x) = (x - 1)^2 - 4$
(b) $f(x) = 2(x + 2)^2 - 2$
(c) $f(x) = 9 - (x - 2)^2$
(d) $f(x) = -2(x - 1)(x - 3)$
(e) $f(x) = -(x + 3)(x + 5)$
(f) $f(x) = 2(x + 1)(x - 3)$
(g) $f(x) = -x^2 + 4x + 5$
(h) $f(x) = 2x^2 + 3x - 2$
(i) $f(x) = -x^2 + 4x + 12$

Solving problems of quadratic functions

The knowledge on quadratic function is very important and it is widely used in our daily lives. The graph of a quadratic function in the shape of a parabola helps us to solve many problems. For example, it can be used to forecast the profit or loss in business, plot the curved motion of an object and determine the minimum or maximum values.

Example 16 MATHEMATICS APPLICATION

Suresh is chosen to represent his school in the district level javelin competition. He throws the javelin at a distance of 3 metres from the ground. The height of the javelin is given by the function $h(t) = -5t^2 + 14t + 3$, where h is the height, in metres, of the javelin and t is the time, in seconds.

(a) Find the maximum height, in metres, of the javelin thrown by Suresh.
(b) Calculate the time, in seconds, when the javelin touches the ground.

Solution

1. Understanding the problems

The function of the height of javelin is $h(t) = -5t^2 + 14t + 3$, where h is the height of javelin, in metres, and t is the time, in seconds, after the javelin is thrown.

2. Planning a strategy

- Express the quadratic function in vertex form and determine the maximum value.
- Solve the equation $h(t) = 0$ to find the intercept on the t-axis, that is the time for the javelin to touch the ground.

2.3.5 2.3.6

3. Implementing the strategy

(a) $h(t) = -5t^2 + 14t + 3$

$= -5\left(t^2 - \frac{14}{5}t - \frac{3}{5}\right)$ ← Make the coefficient of t^2 as 1

$= -5\left(t^2 - \frac{14}{5}t + \left(-\frac{7}{5}\right)^2 - \left(-\frac{7}{5}\right)^2 - \frac{3}{5}\right)$ ← Add and subtract $\left(\frac{\text{coefficient of } t}{2}\right)^2$

$= -5\left[\left(t - \frac{7}{5}\right)^2 - \frac{64}{25}\right]$

$= -5\left(t - \frac{7}{5}\right)^2 + \frac{64}{5}$ ← Vertex is $\left(\frac{7}{5}, \frac{64}{5}\right)$

Since $a < 0$, then the maximum value of $h(t)$ is $\frac{64}{5}$ when $t = \frac{7}{5}$.

Thus, the maximum height reached by the javelin is $\frac{64}{5}$ metres = 12.8 metres.

(b) $h(t) = 0$

$-5t + 14t + 3 = 0$

$5t^2 - 14t - 3 = 0$

$(5t + 1)(t - 3) = 0$

$t = -\frac{1}{5}$ (rejected) or $t = 3$

Thus, the time when the javelin touches the ground is 3 seconds.

4. Making a conclusion

Function $h(t) = -5t^2 + 14t + 3$.

(a) Coordinates of the maximum height:

$t = -\frac{b}{2a}$

$= -\frac{14}{2(-5)}$

$= 1.4$

Substitute $t = 1.4$ into the quadratic function,

$h(1.4) = -5(1.4)^2 + 14(1.4) + 3$

$= 12.8$

Thus, the maximum height achieved by the javelin is 12.8 metres after 1.4 seconds.

(b) When the time is 3 seconds:

$h(t) = -5(3)^2 + 14(3) + 3$

$= -45 + 42 + 3$

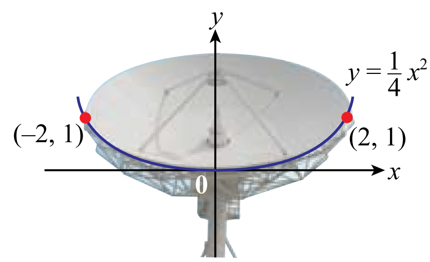
$= 0$

Self Practice 2.11

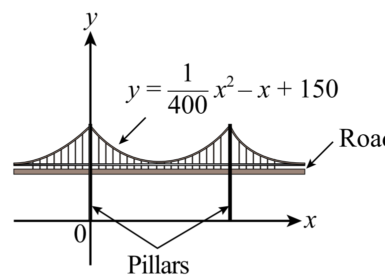
1. The function $h(t) = -5t^2 + 8t + 4$ represents the height h, in metres, of a diver from the water surface in a swimming pool, t seconds after he dives from the diving board. Find
   (a) the height, in metres, of the diving board from the water surface,
   (b) the time, in seconds, the diver achieves at the maximum height,
   (c) the maximum height, in metres, the diver achieves,
   (d) the time, in seconds, when the diver is in the air.

2. A tunnel at a certain part of a highway is in the shape of parabola. The height, in metres, of the curve of the parabola is given by the function $h(x) = 15 - 0.06x^2$, where x is the width of the tunnel, in metres.
   (a) Determine the maximum height, in metres, of the tunnel.
   (b) Find the width, in metres, of the tunnel.

3. The diagram on the right shows the cross section of a parabolic satellite whose function is represented by $f(x) = \frac{1}{4}x^2$, where x and y are measured in metres. Find the width and the depth of the parabola, in metres.



4. The diagram on the right shows a suspension bridge. The function of the cables between the two pillars of the bridge is represented by $y = \frac{1}{400}x^2 - x + 150$, where x and y are measured in metres. The minimum point of the cable is on the road surface at the middle of the two pillars.
   (a) What is the distance between the minimum point and each pillar?
   (b) What is the height of the road above the water level?



Intensive Practice 2.3

Scan the QR code or visit bit.ly/33jjtRj for the quiz

1. Find the values or range of values of k, if the quadratic function
   (a) $f(x) = kx^2 - 4x + k - 3$ has only one intercept,
   (b) $f(x) = 3x^2 - 4x - 2(2k + 4)$ intersects the x-axis at two different points.

2. Find the smallest value of integer m such that the function $f(x) = mx^2 + 7x + 3$ is always positive for all real values of x.

3. The quadratic function is defined by $f(x) = x^2 + 6x + n$, where n is a constant.
   (a) Express $f(x)$ in the form $(x - h)^2 + k$, where h and k are constants.
   (b) Given the minimum value of $f(x)$ is –5, find the value of n.
   (c) Sketch the curve of $f(x)$.

2.3.6

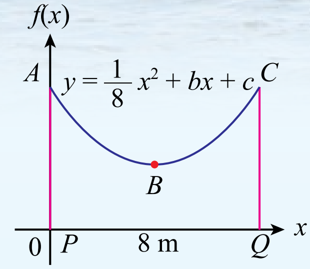
4. Find the range of values of r such that the line $y = rx + 4$ does not intersect the curve $y = x^2 - 4x + 5$. State the values of r such that the line $y = rx + 4$ is a tangent to the curve $y = x^2 - 4x + 5$.

5. Explain the effect on the shape and position of the graph for each change on the following quadratic functions.
(a) Change $f(x) = 3(x - 1)^2 + 2$ to $f(x) = 6(x - 1)^2 + 2$.
(b) Change $f(x) = 3(x - 1)^2 + 2$ to $f(x) = 3(x - 4)^2 + 2$.
(c) Change $f(x) = 3(x - 1)^2 + 2$ to $f(x) = 3(x - 1)^2 + 5$.

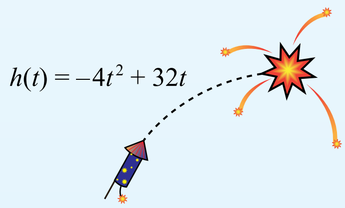
6. The height, h, in metres, of a bird to catch a fish in a pond can be represented by the function $h(t) = 2(t - 3)^2$, where t is the time, in seconds, when the bird starts to fly to catch the fish.
(a) Sketch the graph of $h(t)$.
(b) The movement of another bird is represented by the function $r(t) = 2h(t)$. Sketch the graph of $r(t)$.
(c) Compare the graphs of $h(t)$ and $r(t)$. Which bird starts to move at the highest position? Explain.

7. Given a quadratic function $f(x) = 3 - 4k - (k + 3)x - x^2$, where k is a constant, is always negative when $p < k < q$. Find the value of p and of q.

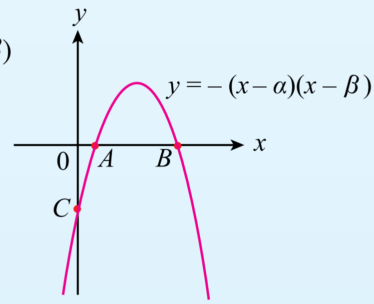
8. The diagram on the right shows a bridge PQ of length 8 m across a river. The supporting cable ABC on the bridge can be represented by the function $f(x) = \frac{1}{8}x^2 + bx + c$, where b and c are constants.
(a) Find the value of b.
(b) Find the value of c such that the minimum point B on the cable is always above PQ.
(c) Find the values of c if b is 2 m above PQ.



9. The function $h(t) = -4t^2 + 32t$ as shown in the diagram on the right represents the height, in metres, of a firework, t seconds after it was launched. The fireworks exploded at the highest point.
(a) When did the firework explode?
(b) What was the height at which the fireworks explode?

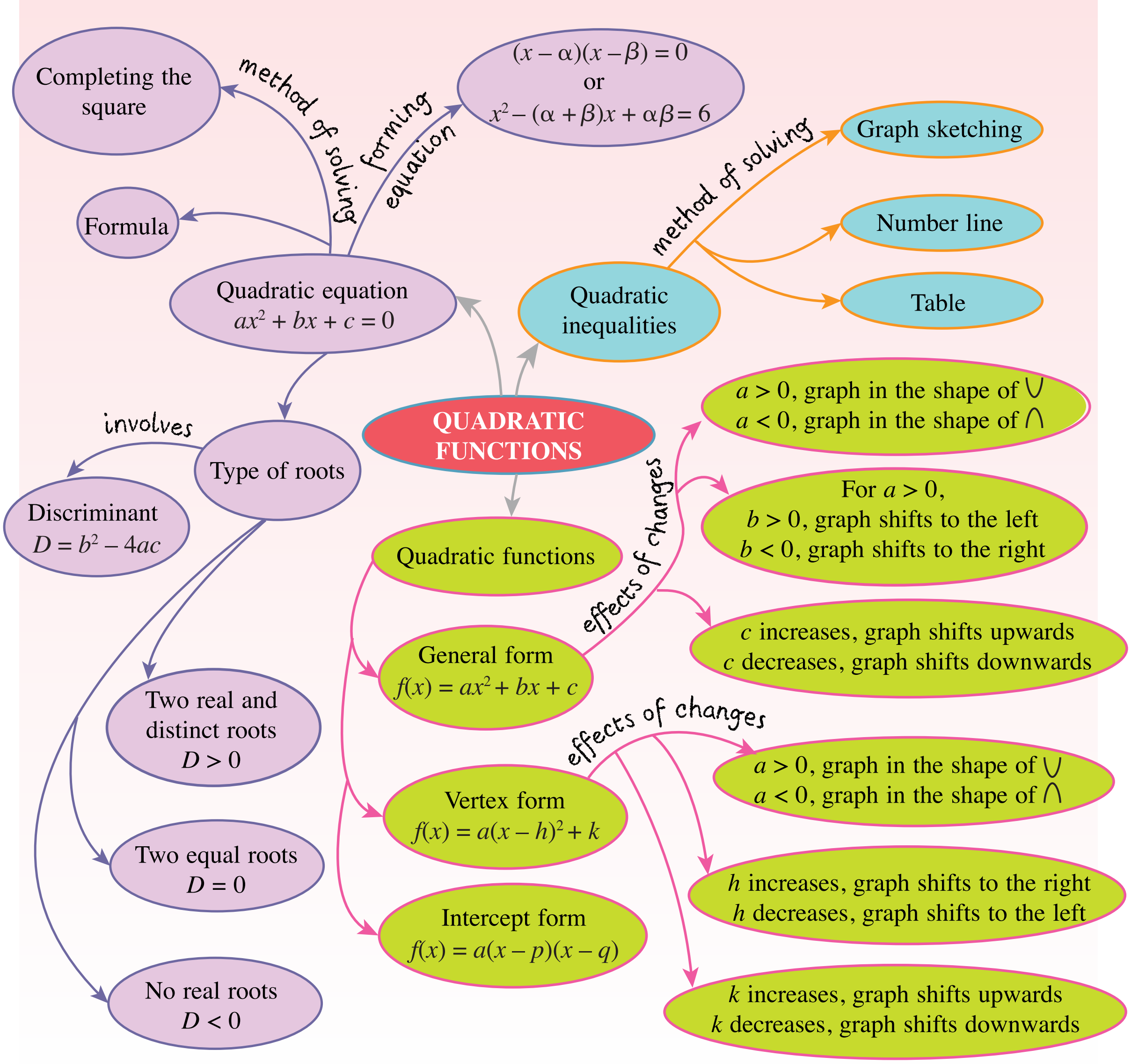


10. The diagram on the right shows the graph of $y = -(x - \alpha)(x - \beta)$ where $\alpha < \beta$.
(a) Given that M is the midpoint of AB, express the following lengths, in terms of α and/or β.
(i) OA (ii) OB
(iii) OC (iv) OM
(b) Can you interpret $\frac{\alpha + \beta}{2}$ and $-\alpha\beta$ in the diagram geometrically?



11. The maximum value of $f(x) = x^2 - 4nx + 5n^2 + 1$ is $m^2 + 2n$ where m and n are constants. Show that $m = n - 1$.

SUMMARY OF CHAPTER 2



WRITE YOUR JOURNAL

The word quadratic originates from the word *quad* which means four but a quadratic equation involves polynomial with the maximum power of 2. Carry out a study on the origin of the word quadratic that is related to quadratic equation. Produce a graphical folio on your study.

MASTERY PRACTICE

1. Solve the quadratic equation $3x(x - 4) = (2 - x)(x + 5)$. Write the answer in three decimal places. PL2

2. Given the quadratic equation $(x - 4)^2 = 3$. PL2
(a) Express the equation in the general form, $ax^2 + bx + c = 0$.
(b) State the sum of roots and products of roots of the equation.
(c) Determine the type of roots of the equation.

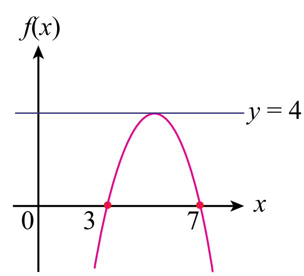
3. Find the values of k or the range of values of k such that the equation $x^2 + kx = k - 8$ has PL2
(a) two equal roots,
(b) two real and different roots,
(c) real roots.

4. Given the quadratic equation $3x^2 + px - 8 = 0$, where p is a constant. Find the value of p if
(a) one of the roots of the equation is -2, PL2
(b) the sum of roots of the equation is $\frac{1}{3}$.

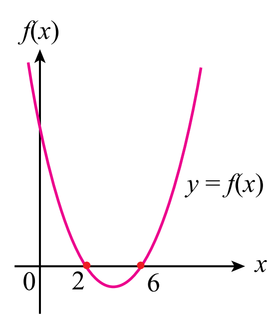
5. Given $3hx^2 - 7kx + 3h = 0$ has two real and equal roots, where h and k are positive. Find the ratio $h : k$ and solve the equation. PL3

6. Find the range of values of x for $x^2 - 7x + 10 > 0$ and $x^2 - 7x \leqslant 0$. Hence, solve the inequality $-10 < x^2 - 7x \leqslant 0$. PL5

7. The diagram on the right shows the graph of quadratic function $f(x) = -\frac{1}{3}[(x + p)^2 + q]$. The line $y = 4$ is the tangent to the curve. Find PL3
(a) the roots for $f(x) = 0$,
(b) the values of p and of q,
(c) the equation of the axis of symmetry of the curve,
(d) the range of values of x when $f(x)$ is positive.



8. The diagram on the right shows the graph of $f(x) = x^2 + bx + c$, where b and c are constants. Find PL3
(a) the values of b and of c,
(b) the coordinates of the minimum point,
(c) the range of values of x when $f(x)$ is negative,
(d) the maximum value when the graph is reflected in the x-axis.



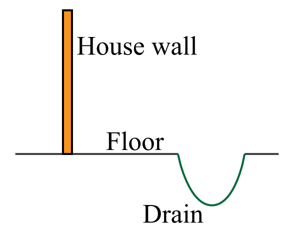
9. A boat moves 24 km to the east and the water current was 3 km/h. The to-and-fro journey took 6 hours. Find the velocity of the boat, in km/h, if the boat maintained its uniform velocity. PL5

10. An ancient Chinese book, Jiuzhang Suanshu which means "Nine Chapters on Mathematics Arts" contained the following problem. PL4

> "The height of a rectangular door is 6.8 units more than its width and the length between two opposite vertices is 100 units. Find the width of the door."

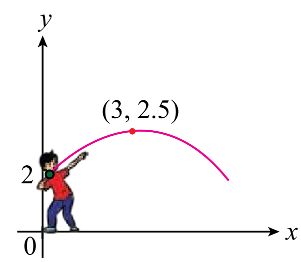
Using a quadratic formula, solve the problem.

11. The diagram on the right shows the cross section of a drain around the house. If the shape of the drain is represented by the equation $y = \frac{1}{5}x^2 - 24x + 700$, find PL5



(a) the width of the opening of the drain,
(b) the minimum depth of the drain.

12. The path of a shot put thrown by Krishna in a competition can be represented by the quadratic function graph as shown in the diagram on the right. The shot put is thrown at a height of 2 m and the path passes through the maximum point (3, 2.5). PL4



(a) Express the equation of the path of shot put in the form $y = a(x - h)^2 + k$, where a, h and k are constants.
(b) Find the maximum distance of the horizontal throw, in metres, by Krishna.

Exploring MATHEMATICS

The functions for three different water spouts in the shape of a parabola in a pool are as follows.

Water spout I: $h = -3d^2 + 4$

Water spout II: $h = -3.5d^2 + 3$

Water spout III: $h = -0.5d^2 + 1$

For each of the functions, h metres represents the height of the water spout and d metres is the horizontal distance of the water spout. Based on the given functions, answer the following questions and explain your reason.
(a) Which water spout emits water from the highest point?
(b) Which water spout follow the narrowest path?
(c) Which water spout has the furthest distance?

CHAPTER 3 Systems of Equations

What will be learnt?

- Systems of Linear Equations in Three Variables
- Simultaneous Equations involving One Linear
- Equation and One Non-Linear Equation

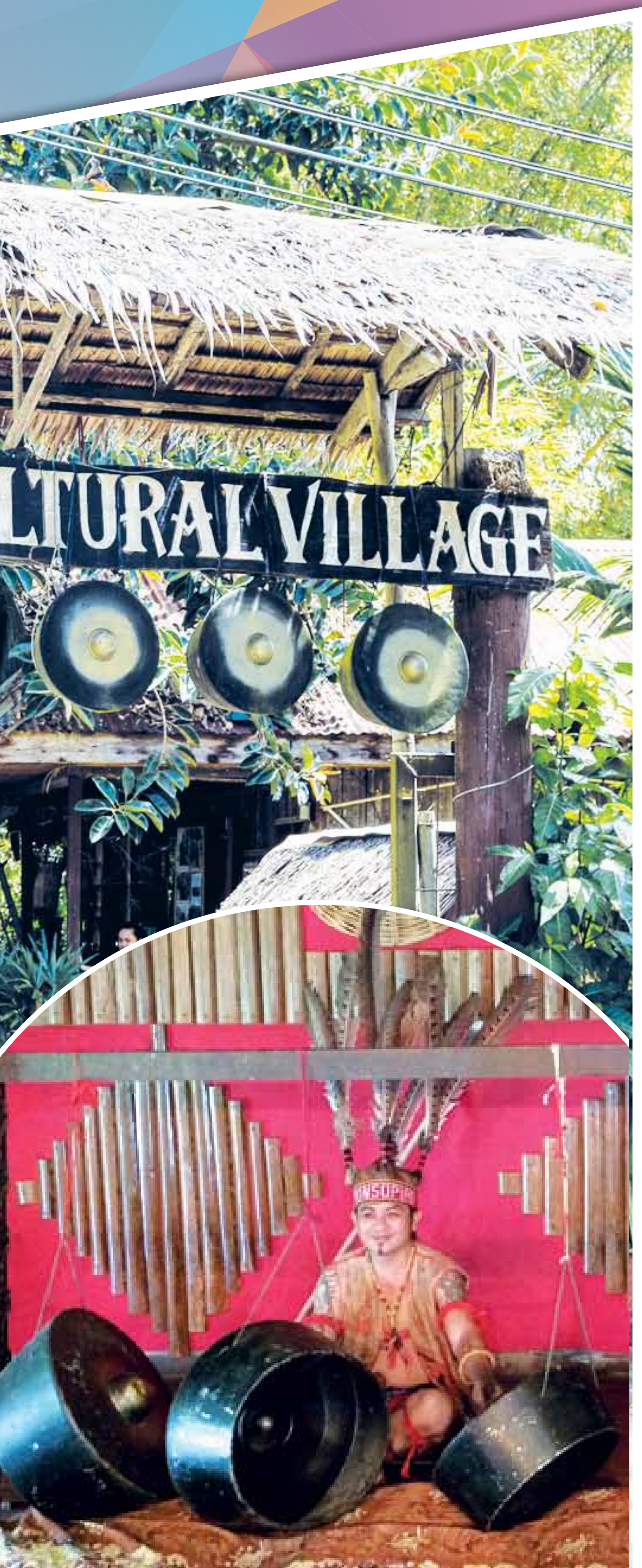
List of Learning Standards

bit.ly/2pWDLl8

KEYWORDS

System of equations	*Sistem persamaan*
Variable	*Pemboleh ubah*
Linear equation	*Persamaan linear*
Non-linear equation	*Persamaan tak linear*
Elimination method	*Kaedah penghapusan*
Substitution method	*Kaedah penggantian*
Graphical method	*Perwakilan graf*

The Monsopiad Cultural Village in Sabah has successfully sold 30 adult tickets to foreign tourists for RM1 100. The ticket price for a trekking package is RM30, whereas the ticket price for an optional package is RM45 and the ticket price for a standard package is RM55. The number of trekking package tickets sold was twice the sum of the optional and standard package tickets. How do you determine the number of tickets sold for each package?



Did you Know?

The solutions to systems of linear equations in three variables can be derived using the Gaussian elimination method. This method was invented circa 1810 by Friedrich Gauss, a German mathematician. This is an alternative method if you do not have a graphical calculator or software.

For further information:

bit.ly/2OCmQi5

SIGNIFICANCE OF THIS CHAPTER

- Engineers use systems of equations to solve problems involving voltages, currents and resistances.
- Biomedical, chemical, electrical, mechanical and nuclear engineers use systems of equations to derive the measurements of solids and liquids.

Scan the QR code to watch the video on the tradisional dance of the Kadazandusun tribe at the Monsopiad Cultural Village.

bit.ly/2FNZjXk

3.1 Systems of Linear Equations in Three Variables

Describing systems of linear equation in three variables

Have a look at the prices of the packages offered by a cinema in the advertisement on the right. How do you determine the price of one ticket, one bottle of drink and one bucket of popcorn?

Three linear equations can be formed using the variables x, y and z to represent the prices of one ticket, one bottle of drink and one bucket of popcorn respectively.

$$x + y + 2z = 13$$
$$2x + 2y + z = 17$$
$$3x + 3y + 2z = 27$$

These linear equations formed are known as systems of linear equations in three variables. A system of linear equations refers to the presence of at least two linear equations which contain the same set of variables. The general form of a linear equation in three variables can be written as follows:

$ax + by + cz = d$, where a, b and c are not equal to zero.

FLASHBACK

A linear equation is an equation where the power of the variable is 1.

Let's see how systems of linear equations in three variables can be expressed in three-dimensional planes.

INQUIRY 1 In groups 21st Century Learning

Aim: To describe systems of linear equations in three variables

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on all three boxes to display all three planes for the equations $x - 3y + z = 4$, $x - 3y + 3z = 4$ and $x - 3y + 3z = 0$.
3. Observe the three planes.
4. Discuss with your groupmates your observations and record your findings in a sheet of paper.
5. Each group shall move to other groups to compare the results obtained.

bit.ly/2ItuOpP

From the results of Inquiry 1, it is found that

> A system of linear equations in three variables has three axes, namely the x-axis, y-axis and z-axis. All three linear equations form a plane on each axis.

INQUIRY 2 In groups 21st Century Learning

Aim: To compare systems of linear equations in two variables and three variables

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on both boxes to display the two straight lines.
3. Observe both lines and record your group's findings in a sheet of paper.
4. Compare your group's findings with the results of Inquiry 1.
5. Present the comparison in front of the class.

bit.ly/35b4J8R

From the results of Inquiry 2, it is found that there are only two axes, namely the x-axis and y-axis.

> Every linear equation in two variables forms a straight line on each axis.

In general, a linear equation in two variables can be written in the form $ax + by = c$, where a, b and c are constants. Geometrically, when a linear equation in two variables is sketched on a plane, it will form a straight line as shown in Diagram 1(a).

Meanwhile, linear equations in three variables can be written in the form $ax + by + cz = d$, where a, b, c and d are constants. When sketched, a three-dimensional plane will be formed, as shown in Diagram 1(b).

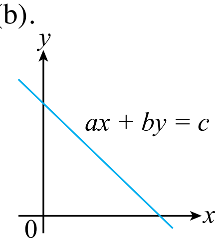


Diagram 1(a)

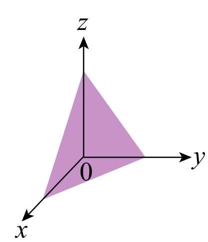


Diagram 1(b)

Example 1

Describe whether the following equations are systems of linear equations in three variables or not.

(a) $2x + 4y - z = 10$
$x + y = 10z^2$
$5y - z - 2x = 3$

(b) $p + 8q - 4r = 2$
$2(p + 6r) + 7q = 0$
$10r + p = 5q$

Solution

(a) No, because there is an equation in which the highest power of the variable is 2.

(b) Yes, because all three equations have three variables, p, q and r, of power 1.

3.1.1

Self Practice 3.1

1. Form linear equations in three variables for the following statement.

Aiman bought 3 pairs of trousers, 2 shirts and a pair of shoes.
He spent RM750 for all his purchases.

2. Explain whether the following equations are systems of linear equations in three variables.

(a) $2m + 6(n - 2p) = 4$
$n = 5m + p$
$4m + p = \dfrac{2m}{5}$

(b) $e(12 - 6g) = f^2$
$8e + 6 - 2f - 9g = 0$
$17f + e = 6 + 2e$

(c) $7a - c = 6b$
$3 - 4c = 10a + b$
$\dfrac{a}{6} + 3b = 2(c + b)$

Solving problems of linear equations in three variables

INQUIRY 3 — In groups

Aim: To solve systems of linear equations in three variables

bit.ly/31Z1gZe

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on all three boxes to display all three planes for the linear equations $2x + y + z = 3$, $-x + 2y + 2z = 1$ and $x - y - 3z = -6$.
3. Do the planes intersect with each other? Determine the point of intersection of its coordinates (x, y, z) between all three planes.
4. Determine whether the point of intersection (x, y, z) is the solution to all three linear equations.
5. Record the opinion of each group member with regards to the relation between the point of intersection and the solution to all the linear equations. Discuss accordingly.

From the results of Inquiry 3, the point of intersection between the three planes is the solution to all three linear equations. In this case, there is only one solution because the planes only intersect at one point.

INQUIRY 4 — In groups

Aim: To solve systems of linear equations in three variables

bit.ly/2Vk8OTC

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on all three boxes to display all three planes for the linear equations $x - 2y = 4$, $2x - 3y + 2z = -2$ and $4x - 7y + 2z = 6$.
3. Do the planes intersect with each other? Determine the point of intersection.
4. Record the opinion of each group member with regards to the relationship between the point of intersection and the solution to all the linear equations. Discuss accordingly.

From the results of Inquiry 4, the three planes intersect along a straight line. Hence, this system of linear equations has infinite solutions.

INQUIRY 5 In groups

Aim: To solve systems of linear equations in three variables

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on all three boxes to display all three planes for the linear equations $2x - 4y + z = 3$, $4x + 8y + 2z = 14$ and $x - 2y + 0.5z = -1$.
3. Are there any intersecting points among the three planes?
4. Record the opinion of each group member and discuss accordingly.

bit.ly/2loqoRh

From the results of Inquiry 5, it is found that the planes of all three linear equations do not intersect at any point. Hence, this system of linear equations has no solution.

The results of Inquiry 3, 4 and 5 show that there are three types of solutions to systems of linear equations in three variables as shown in the diagram below.

BRAINSTORMING

Using the *GeoGebra* software, determine the systems of linear equations in three variables which have
(a) only one solution
(b) infinite solutions

Solutions to systems of linear equations in three variables

One solution	Infinite solutions	No solution
The planes intersect at only one point	The planes intersect along a straight line	The planes do not intersect at any point

FLASHBACK

Simultaneous equations in two variables can be solved using the graphical method, substitution method or elimination method.

Systems of linear equations in three variables can be solved by finding the values of all variables so that they satisfy all three linear equations. The methods for solving systems of linear equations in three variables are the elimination and substitution methods.

The steps for solving systems of linear equations in three variables through the elimination and substitution methods are similar to the method for solving simultaneous equations in two variables.

FLASHBACK

There are three types of solutions to simultaneous equations in two variables. When both lines are:

- Intersecting, there is a unique solution.
- Parallel, there is no solution.
- Overlapping, there are infinite solutions.

Example 2

Solve the following system of linear equations using the elimination method.

$$4x - 3y + z = -10$$
$$2x + y + 3z = 0$$
$$-x + 2y - 5z = 17$$

Solution

Choose any two equations.

$4x - 3y + z = -10$ ··· ①
$2x + y + 3z = 0$ ··· ②

Multiply equation ② with 2 so that the coefficients of x are equal.

② × 2: $4x + 2y + 6z = 0$ ··· ③

Eliminate the variable x by subtracting ① from ③.

③ – ①: $5y + 5z = 10$ ··· ④

Choose another set of two equations.

$2x + y + 3z = 0$ ··· ⑤
$-x + 2y - 5z = 17$ ··· ⑥

Multiply equation ⑥ with 2 so that the coefficients of variable x are equal.

⑥ × 2: $-2x + 4y - 10z = 34$ ··· ⑦
⑤ + ⑦: $5y - 7z = 34$ ··· ⑧
④ – ⑧: $12z = -24$
$z = -2$

Substitute $z = -2$ into ⑧.

$5y - 7(-2) = 34$
$5y + 14 = 34$
$5y = 20$
$y = 4$

Substitute $y = 4$ and $z = -2$ into ①.

$4x - 3(4) + (-2) = -10$
$4x - 12 - 2 = -10$
$4x = 4$
$x = 1$

Thus, $x = 1$, $y = 4$ and $z = -2$ are the solutions to this system of linear equations.

QUICK COUNT

Determine the solution to Example 2 using a scientific calculator.

1. Press [MENU]
2. Press [▼] twice
3. Press [ALPHA] [(–)]
4. Press [1] for Simul Equation
5. Press [3]
6. Key in the coefficients of x, y and z.
   Press [4] [=], [–3] [=], [1] [=], [–10] [=], [2] [=], [1] [=], [3] [=], [0] [=], [–1] [=], [2] [=], [–5] [=], [17] [=]
7. The screen will show:
   x 1
   Press [=]
   y 4
   Press [=]
   z –2

Thus, $x = 1$, $y = 4$ and $z = -2$.

Example 3

Solve the following system of linear equations using the substitution method.

$$3x - y - z = -120$$
$$y - 2z = 30$$
$$x + y + z = 180$$

Solution

$3x - y - z = -120 \quad \cdots ①$
$y - 2z = 30 \quad \cdots ②$
$x + y + z = 180 \quad \cdots ③$

From ①, $z = 3x - y + 120 \quad \cdots ④$ ← Express z in terms of x and y

Substitute ④ into ②.

$y - 2(3x - y + 120) = 30$
$y - 6x + 2y - 240 = 30$
$-6x + 3y = 270$
$y = 90 + 2x \quad \cdots ⑤$ ← Express y in terms of x

Substitute ④ and ⑤ into ③.

$x + (90 + 2x) + [3x - (90 + 2x) + 120] = 180$
$x + 2x + 3x - 2x + 90 - 90 + 120 = 180$
$4x = 60$
$x = 15$

Substitute $x = 15$ into ⑤.

$y = 90 + 2(15)$
$= 120$

Substitute $x = 15$ and $y = 120$ into ③.

$15 + 120 + z = 180$
$z = 45$

Thus, $x = 15$, $y = 120$ and $z = 45$ are the solutions to this system of linear equations.

QR

The Gaussian elimination method can also be used to solve systems of linear equations. Visit the following link for more information on the Gaussian elimination method.

bit.ly/2nA2g6U

Mind Challenge

Solve Example 3 using the elimination method. Did you get the same solution?

CHAPTER 3

Example 4

Solve the following system of linear equations.

$x - y + 3z = 3$
$-2x + 2y - 6z = 6$
$y - 5z = -3$

Solution

$x - y + 3z = 3 \quad \cdots ①$
$-2x + 2y - 6z = 6 \quad \cdots ②$
$y - 5z = -3 \quad \cdots ③$

Since equation ③ only has two variables, which are y and z, then the variable x in equations ① and ② need to be eliminated.

① × 2: $2x - 2y + 6z = 6 \quad \cdots ④$
④ + ②: $0 + 0 + 0 = 12$
$0 = 12$

Thus, this system of linear equations has no solution as $0 \neq 12$.

QR

Solution to Example 4 using the *GeoGebra* software.

bit.ly/2OwUo0W

3.1.2

Example 5

Solve the following system of linear equations.

$$3x + 5y - 2z = 13$$
$$-5x - 2y - 4z = 20$$
$$-14x - 17y + 2z = -19$$

Solution

$3x + 5y - 2z = 13$ ··· ①
$-5x - 2y - 4z = 20$ ··· ②
$-14x - 17y + 2z = -19$ ··· ③
① × 2: $6x + 10y - 4z = 26$ ··· ④
④ – ②: $11x + 12y = 6$ ··· ⑤
① + ③: $-11x - 12y = -6$ ··· ⑥
⑤ + ⑥: $0 + 0 = 0$
$0 = 0$

Multiply equation ① with 2 to eliminate the variable z

Thus, this system of linear equations has infinite solutions because $0 = 0$.

Self Practice 3.2

1. Solve the following systems of linear equations using the elimination method.

(a) $7x + 5y - 3z = 16$
$3x - 5y + 2z = -8$
$5x + 3y - 7z = 0$

(b) $4x - 2y + 3z = 1$
$x + 3y - 4z = -7$
$3x + y + 2z = 5$

2. Solve the following systems of linear equations using the substitution method.

(a) $2x + y + 3z = -2$
$x - y - z = -3$
$3x - 2y + 3z = -12$

(b) $2x + 3y + 2z = 16$
$x + 4y - 2z = 12$
$x + y + 4z = 20$

Solving problems involving systems of linear equations in three variables

Example 6

MATHEMATICS APPLICATION

Tommy has three condominium units, type A with 1 bedroom, type B with 2 bedrooms and type C with 3 bedrooms. All three condominium units are rented and the total rent received by Tommy is RM1 240 per day. Tommy needs to save 10% of the rent of the type A unit, 20% of the rent of the type B unit, and 30% of the rent of the type C unit to pay for maintenance charges. The total daily savings is RM276. The rent for the type C unit is twice the rent for the type A unit. What is the daily rent for each condominium unit owned by Tommy?

3.1.2 3.1.3

Solution

1. Understanding the problem

- Total rent is RM1 240 per day.
- Savings for maintenance charges:
  - Type A unit is 10% of the rent.
  - Type B unit is 20% of the rent.
  - Type C unit is 30% of the rent.
- Total daily savings in RM276.
- The rent for the type C unit is twice the rent for the type A unit.

2. Planning a strategy

- Form three equations which involve the total daily rent, total daily savings and the rent of the type C unit.
- Let the rent of the type A unit be a, rent of the type B unit be b and rent of the type C unit be c.

3. Implementing the strategy

$a + b + c = 1\ 240$ ⋯ ①
$0.1a + 0.2b + 0.3c = 276$ ⋯ ②
$c = 2a$ ⋯ ③
② × 10: $a + 2b + 3c = 2\ 760$ ⋯ ④
① × 2: $2a + 2b + 2c = 2\ 480$ ⋯ ⑤
④ − ⑤ : $-a + c = 280$ ⋯ ⑥

Substitute ③ into ⑥.
$-a + 2a = 280$
$a = 280$

Substitute $a = 280$ into ③.
$c = 2(280)$
$= 560$

Substitute $a = 280$ and $c = 560$ into ①.
$280 + b + 560 = 1\ 240$
$840 + b = 1\ 240$
$b = 400$

The rent for the type A condominium unit is RM280, type B condominium unit RM400 and type C condominium unit RM560.

4. Making a conclusion

Total rent per day
$= 280 + 400 + 560$
$=$ RM 1 240
Total daily savings
$= 0.1(280) + 0.2(400) + 0.3(560)$
$= 28 + 80 + 168$
$=$ RM276

Self Practice 3.3

1. Patricia invested RM24 500 in three unit trusts. She divided the money into three different unit trust accounts, P, Q and R. At the end of the year, she obtained a profit of RM1 300. The annual dividends for the accounts are 4%, 5.5% and 6% respectively. The final amount of money in account P is four times that in account Q. How much money did she invest in each unit trust account?

2. Billy Restaurant ordered 200 stalks of flowers in conjunction with Mother's Day. They ordered carnations which cost RM1.50 per stalk, roses which cost RM5.75 per stalk and daisies which cost RM2.60 per stalk. Of the three types of flowers ordered, the number of carnations was the largest. Meanwhile, the number of roses was 20 stalks less than that of daisies. The total price of the flowers is RM589.50. How many flowers of each type were ordered?

3.1.3

3. Ramasamy wants to buy pens, pencils and notebooks for the new school term. He has RM102 to spend. The price of a pen is RM5, a pencil RM3, and a notebook RM9. Ramasamy intends to spend the same amount of money on pens and pencils. The total number of pens and pencils to be purchased must be equal to the number of notebooks to be purchased. How many of each item will he purchase? Write a system of equations to solve this problem.

Intensive Practice 3.1

Scan the QR code or visit bit.ly/2IEEJZT for the quiz

1. Form systems of linear equations for the following situations and answer the questions given.
   (a) The sum of angles in a triangle is 180°. The largest angle is 20° more than the sum of the other two angles and is 10° more than three times the smallest angle. What is the measure of each angle of the triangle?
   (b) The sum of three numbers is 19. If the first number is multiplied by 2, the sum of the three numbers is 22, and if the second number is multiplied by 2, the sum becomes 25. Find the value of each number.

2. Solve the following equations using the elimination and substitution methods.

(a) $x + y + z = 3$ $x + z = 2$ $2x + y + z = 5$	(b) $2x + y - z = 7$ $x - y + z = 2$ $x + y - 3z = 2$	(c) $x + y + z = 3$ $2x + y - z = 6$ $x + 2y + 3z = 2$
(d) $2x - y + z = 6$ $3x + y - z = 2$ $x + 2y - 4z = 8$	(e) $x + y + 2z = 4$ $x + y + 3z = 5$ $2x + y + z = 2$	(f) $x + 2y + z = 4$ $x - y + z = 1$ $2x + y + 2z = 2$

3. A bakery bakes three types of bread, with the monthly cost being RM6 850 for 2 150 loaves of bread. The cost of baking a loaf of butterscotch bread is RM2, a chocolate bread RM3 and a coconut bread RM4. The sale prices of a loaf of butterscotch bread, a chocolate bread and a coconut bread are RM3, RM4.50 and RM5.50 respectively. If the bakery makes a profit of RM2 975 monthly, how many loaves of each type of bread will it bake?

4. Andrea sells flower pots of different sizes. The price of a small flower pot is RM10, medium flower pot RM15 and large flower pot RM40. Every month, the number of small flower pots sold is equal to the total number of the medium and large flower pots sold. The number of medium flower pots sold is twice the number of large flower pots sold. Andrea needs to pay a rent of RM300 per month for her business premise. What are the minimum numbers of pots of each size which Andrea has to sell in a month so that she can pay the monthly rent?

5. Mr Chong intends to purchase a few chickens, rabbits and ducks for his farm. The total number of animals is 50. He has RM1 500 to be spent. A chicken costs RM20, a rabbit RM50 and a duck RM30. The number of chickens is equal to the number of ducks. How many of each animal should Mr Chong purchase? Write a system of linear equations to solve this problem.

3.2 Simultaneous Equations Involving One Linear Equation and One Non-Linear Equation

Solving simultaneous equations involving one linear equation and one non-linear equation

INQUIRY 6 In groups 21st Century Learning

Aim: To identify simultaneous equations

Instructions:

1. Form a few groups of three to four members each.
2. Read the following statements and construct the appropriate equations.

STATEMENT 1
Chong has a rectangular garden. The length of fencing which will be used to fence the garden is 200 m. The area of the garden is 2 400 m^2. What is the length and width of the garden?

STATEMENT 2
Shida is sewing a rectangular tablecloth. The perimeter of the tablecloth is 800 cm and the area is 30 000 cm^2. Find the length and width of the table.

STATEMENT 3
The difference between two numbers is 9 and the product of the two numbers is 96. Find the values of the numbers.

3. Answer the following questions:
   (a) How many equations can be formed in each statement?
   (b) How many variables are involved?
4. Discuss with your group members and record your findings on a piece of paper.
5. Each group shall elect a representative to present their findings to the class.
6. Other group members can ask questions to the elected representatives.
7. Repeat steps 5 and 6 until all groups have finished presenting.

From the results of Inquiry 6, each of the three statements had two equations of two variables, namely a linear equation and non-linear equation. What are the characteristics which distinguish linear equations from non-linear equations? How do we solve simultaneous equations which involve a linear equation and non-linear equation?

3.2.1

INQUIRY 7 Individual 21st Century Learning

Aim: To explore the point of intersection between linear and non-linear equations

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on both boxes to display the shapes of graphs for the equations $x + 2y = 10$ and $y^2 + 4x = 50$.
3. What can you conclude about the shapes of the two graphs?

bit.ly/2OqKBK7

From the results of Inquiry 7, the point of intersection between the graph of linear equation $x + 2y = 10$ and the non-linear equation $y^2 + 4x = 50$ is the solution to both equations. The solution to both equations is also known as the solution of simultaneous equations.

Solving simultaneous equations means finding the values of the variables which satisfy the equations concerned. These simultaneous equations can be solved using the elimination method, substitution method or graphical representation method.

Example 7

Solve the following simultaneous equations using the substitution method.

$$2x + y = 4$$
$$y^2 + 5 = 4x$$

Solution

$$2x + y = 4 \quad \cdots ①$$
$$y^2 + 5 = 4x \quad \cdots ②$$

From ①,

$$2x = 4 - y$$
$$x = \frac{4 - y}{2} \quad \cdots ③$$

Make x the subject of the formula

Substitute ③ into ②.

$$y^2 + 5 = 4\left(\frac{4 - y}{2}\right)$$
$$y^2 + 5 = 8 - 2y$$
$$y^2 + 2y - 3 = 0$$
$$(y + 3)(y - 1) = 0$$
$$y = -3 \text{ or } y = 1$$

Solve the quadratic equation by factorisation

Substitute $y = -3$ and $y = 1$ into ③.

$$x = \frac{4 - (-3)}{2} = \frac{7}{2} \quad \text{or} \quad x = \frac{4 - 1}{2} = \frac{3}{2}$$

Thus, $x = \frac{7}{2}$, $y = -3$ and $x = \frac{3}{2}$, $y = 1$ are the solutions to these simultaneous equations.

Mind Challenge

Solve Example 7 when y is expressed in terms of x for the linear equation $2x + y = 4$. Did you get the same solutions?

FLASHBACK

Quadratic equations can be solved using the following methods:
(a) Factorisation
(b) Formula
(c) Completing the square

Example 8

Solve the following simultaneous equations using the elimination method.

$$2x + y = 4$$
$$x^2 - 2xy = 3$$

Solution

$$2x + y = 4 \quad \cdots ①$$
$$x^2 - 2xy = 3 \quad \cdots ②$$
$$① \times 2x: 4x^2 + 2xy = 8x \quad \cdots ③$$
$$② + ③: \quad 5x^2 = 3 + 8x$$
$$5x^2 - 8x - 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula

$$= \frac{8 \pm \sqrt{(-8)^2 - 4(5)(-3)}}{2(5)}$$
$$x = 1.9136 \quad \text{or} \quad x = -0.3136$$

Substitute $x = 1.9136$ into ①.

$$2(1.9136) + y = 4$$
$$3.8272 + y = 4$$
$$y = 0.1728$$

Substitute $x = -0.3136$ into ①.

$$2(-0.3136) + y = 4$$
$$-0.6272 + y = 4$$
$$y = 4.6272$$

Thus, $x = 1.9136$, $y = 0.1728$ and $x = -0.3136$, $y = 4.6272$ are the solutions to these simultaneous equations.

Example 9

Solve the following simultaneous equations using the graphical representation method.

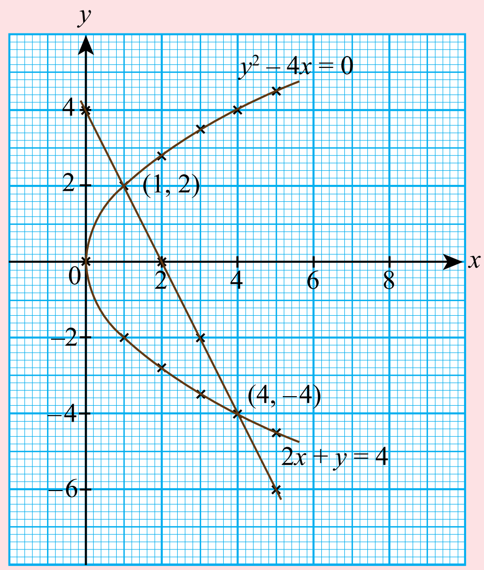
$$2x + y = 4$$
$$y^2 - 4x = 0$$

Solution

Construct a table to determine the points which need to be plotted.

x	0	1	2	3	4	5
Value of y in the equation $2x + y = 4$	4	2	0	–2	–4	–6
Value of y in the equation $y^2 - 4x = 0$	0	±2	±2.8	±3.5	±4	±4.5

Construct a graph based on the values in the table.



Based on the graph above, there are two points of intersection which represent the solutions to both equations. Thus, the solutions to these simultaneous equations are (1, 2) and (4, –4).

Mind Challenge

Solve the following simultaneous equations $2x + y = x^2 - xy + y^2 = 7$.

Tech Whizz

The solution to Example 9 using the *Desmos* software

bit.ly/2KHpcbw

Self Practice 3.4

1. Solve the following simultaneous equations using the elimination, substitution or graphical representation method.

(a) $2x - y = 7$
$y^2 - x(x + y) = 11$

(b) $5y + x = 1$
$x + 3y^2 = -1$

(c) $y = 3 - x$
$\frac{1}{x} - \frac{1}{y} = 2$

(d) $3x + 5y = 1$
$x + 2y = \frac{4}{y}$

(e) $2x + 4y = 9$
$4x^2 + 16y^2 = 20x + 4y - 19$

(f) $x + y - 4 = 0$
$x^2 - y^2 - 2xy = 2$

2. Solve the following simultaneous equations using the graphical representation method.

(a) Draw the graph for the following pairs of equations for the domain $-5 \leqslant x \leqslant 5$. Hence, determine the solution to the simultaneous equations.

$$2y - x = 1$$
$$xy + x^2 = 26$$

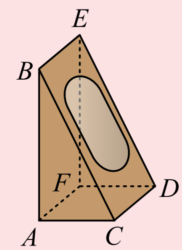
(b) Draw the graph for the following pairs of equations for the domain $-3 \leqslant x \leqslant 4$. Hence, determine the solution to the simultaneous equations.

$$x - y = 2$$
$$4x^2 + 3y^2 = 36$$

Solving problems involving simultaneous equations

Example 10 MATHEMATICS APPLICATION

A food packaging factory intends to pack *dodol* in a container in the shape of a right prism with a square base, as shown in the diagram. Given that the total length of sides of the right prism is 133 cm and $ED = BC = 25$ cm. Can a piece of *dodol* of volume 600 cm^3 be packed into the container? Justify your answer.



Solution

1. Understanding the problem

- Container in the shape of a right prism with a square base.
- Total length of sides of the container = 133 cm
- $ED = BC = 25$ cm
- To determine whether *dodol* of volume 600 cm^3 can be packed into the container.

2. Planning a strategy

- Let the length of side of the base be x and the height be y.
- Form a non-linear equation for the length of BC.
- Form a linear equation for the total length of sides of the prism.
- Volume of prism = area of cross-section × height

3. Implementing the strategy

$$x^2 + y^2 = 25^2 \quad \cdots ①$$
$$5x + 2y + 50 = 133$$
$$5x + 2y = 83 \quad \cdots ②$$

From ②,

$$y = \frac{83 - 5x}{2} \quad \cdots ③$$

Substitute ③ into ①.

$$x^2 + \left(\frac{83 - 5x}{2}\right)^2 = 25^2$$
$$x^2 + \left(\frac{6\,889 - 830x + 25x^2}{4}\right) = 625$$
$$4x^2 + 25x^2 - 830x + 6\,889 - 2\,500 = 0$$
$$29x^2 - 830x + 4\,389 = 0$$
$$(29x - 627)(x - 7) = 0$$
$$x = \frac{627}{29} \quad \text{or} \quad x = 7$$

Substitute $x = \frac{627}{29}$ into ③.

$$y = \frac{83 - 5\left(\frac{627}{29}\right)}{2} = -\frac{364}{29} \text{ (Ignored)}$$

Substitute $x = 7$ into ③.

$$y = \frac{83 - 5(7)}{2} = 24$$

Volume of container $= \frac{1}{2} \times 7 \times 7 \times 24 = 588 \text{ cm}^3$

Hence, a piece of dodol of volume 600 cm^3 cannot be packed into the container because the volume of container is only 588 cm^3.

4. Making a conclusion

Volume of container = 588

$$\frac{1}{2} \times 7 \times 24 \times x = 588$$
$$x = 7 \text{ cm}$$

Substitute $x = 7$ into equation ②

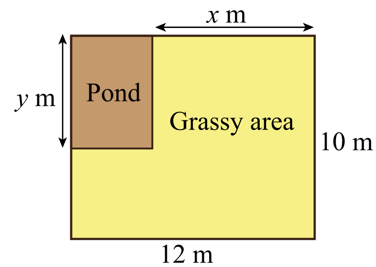
$$5(7) + 2y = 83$$
$$y = 24 \text{ cm}$$

3.2.2

Self Practice 3.5

1. Audy cuts out a rectangular plank of area 72 cm^2 and perimeter 34 cm. Calculate the length and width of the plank.

2. The diagram shows the plan of a rectangular garden which will be constructed by Syarikat Pesona Alam. The corner of a garden will have a rectangular pond. The area to be covered with grass is 96 m^2 and the perimeter of the pond is 20 m. Calculate the value of x and of y.



Intensive Practice 3.2

Scan the QR code or visit bit.ly/2IDhbEE for the quiz

1. Solve the following simultaneous equations.
   (a) $x - 3y + 4 = 0$
   $x^2 + xy - 40 = 0$
   (b) $k - 3p = -1$
   $p + pk - 2k = 0$

2. Find the coordinates of the points of intersection of the curve $\frac{x}{y} - \frac{2y}{x} = 1$ and straight line $2x + y = 3$.

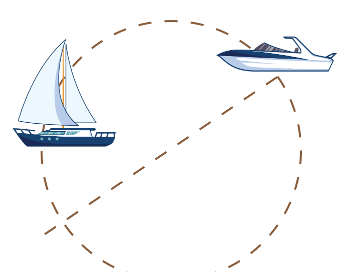
3. Given that $(-2, 2)$ is the solution to the following simultaneous equations:
$$x + \frac{1}{2}y = \frac{h}{2} \text{ and } \frac{1}{x} + \frac{2}{y} = k$$
Find the value of h and of k. Hence, find another solution.

4. The hypothenuse of a right-angled triangle is $(2x + 3)$ cm. The lengths of two other sides are x cm and $(x + y)$ cm respectively. Given that the perimeter of the triangle is 30 cm, find the value of x and of y.

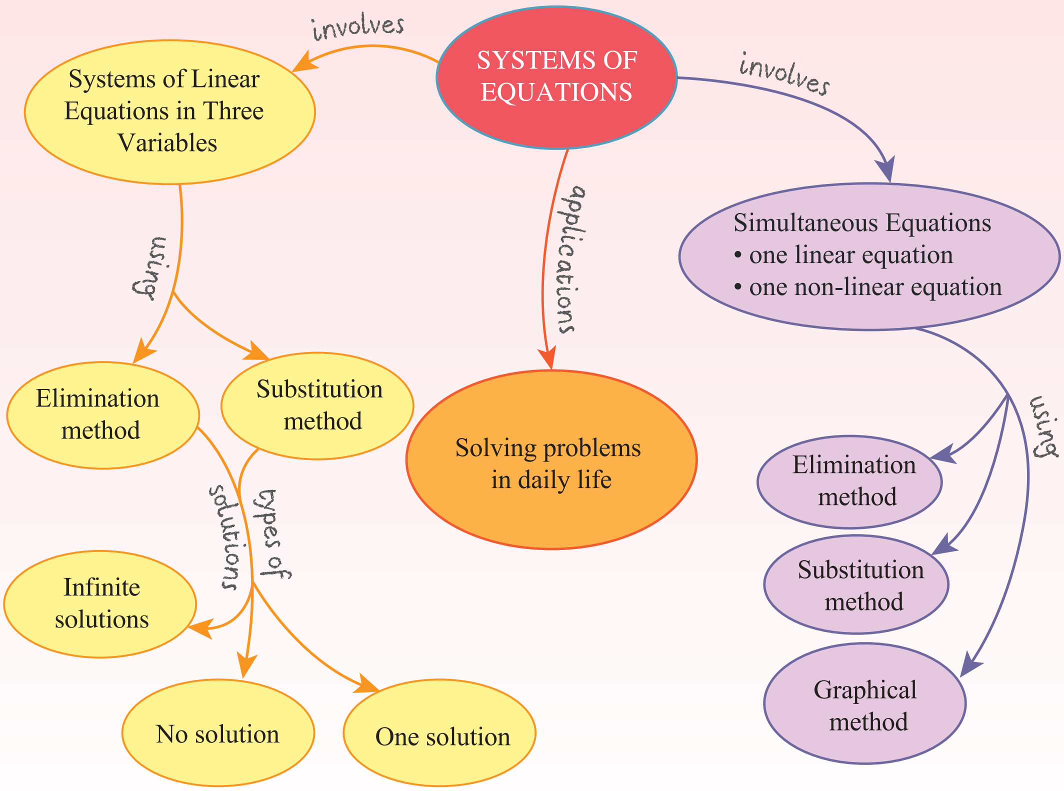
5. Given that the total surface area of a cuboid with a square base is 66 cm^2 and the total length of the sides is 40 cm. Find the possible volumes of the cuboid.

6. A fish is moving in a circular manner, with the equation of its locus being $2x^2 + 11y^2 + 2x + 2y = 0$. A boat is moving along a straight line of equation $x - 3y + 1 = 0$ and it intersects with the circular locus of the fish. Find the points of intersection between the locus of the fish and trajectory of the boat.

7. A sailing boat is moving in a circular manner, with the equation of its locus being $2x^2 + 4y^2 + 3x - 5y = 25$. Meanwhile, a speedboat is moving along a straight line of equation $y - x + 1 = 0$ and it intersects with the locus of the sailing boat. Find the points of intersection between the trajectories of the sailing boat and the speedboat.



SUMMARY OF CHAPTER 3



WRITE YOUR JOURNAL

Think of a problem in your surroundings which can be solved using systems of linear and non-linear equations. Formulate the problem in the form of a system of linear equations with proper definitions for the variables used. State the relation between the variables. Solve the system of equations which has been constructed. Then, write a report on the problem and display it to the class.

MASTERY PRACTICE

1. Form systems of linear equations in three variables for the following situations. PL1

(a) Abdullah buys a History book, two Mathematics books and three Science books for RM120. Chong buys two History books, three Mathematics books and two Science books for RM110. Meanwhile, Kaladevie buys a History book, four Mathematics books and two Science books for RM180.

(b) There are a total of 30 coins which consist of 10 cents, 20 cents and 50 cents coins in a box. The total value of the coins is RM20.60. Salmah bought an ice cream using two 50 cents and three 20 cents coins.

2. Solve the following systems of linear equations. PL2

(a) $x - y + 2z = 3$
$x + y - 3z = -10$
$2x + y - z = -6$

(b) $x + 2y + 5z = -17$
$2x - 3y + 2z = -16$
$3x + y - z = 3$

3. The second angle of a triangle is 50° less than four times the first angle. The third angle is 40° less than the first angle. Find the value of each angle in the triangle. PL3

4. Given that $(5, h)$ is one of the solutions to the following simultaneous equations. PL4

$$h(x - y) = x + y - 1 = hx^2 - 11y^2$$

Find the value of h and the other solution to the simultaneous equations.

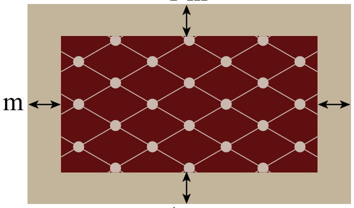
5. Every month, Raju receives sources of income from his fixed salary as a sales officer, house rental and online sales. His total monthly salary is RM20 000. If RM500 is added to his monthly salary, it will be twice the total income from house rental and online sales. The total monthly salary and online sales income is twice the house rental income. How much does Raju receive from each source of income every month? PL4

6. Encik Abu plants vegetables on a plot of land in the shape of a right-angled triangle. Given that the longest side of the land is p metres, the other two sides are q metres and $(2q - 1)$ metres respectively. Encik Abu fenced the land using a fencing of length 40 metres. Find the length, in metres, of each side of the land. PL4

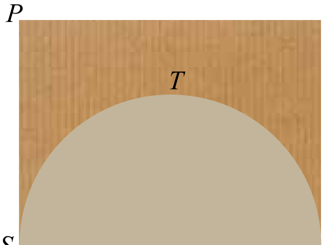
7. Prove that a straight line passing through $(0, -3)$ intersects a curve $x^2 + y^2 - 27x + 41 = 0$ at point $(2, 3)$. Does the straight line intersect the curve at any other point? Justify your answer. PL4

8. A piece of wood measures y cm in length and $3x$ cm in width. A worker intends to cut the piece of wood into two small triangular pieces of wood. The perimeter of each triangle is 24 cm and the longest side of either triangle is $(x + y)$ cm. Calculate the area, in cm^2, of the original piece of wood. PL4

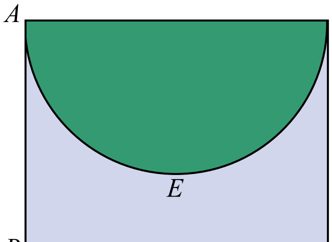
9. The diagram on the right shows the plan of a rectangular room. A rectangular carpet is to be placed with a distance of 1 m from the wall of the room. The area and perimeter of the carpet are 8.75 m^2 and 12 m respectively. Find the length and width, in m, of the room. PL4



10. The diagram on the right shows a rectangular piece of cardboard *PQRS* of area 224 cm^2. A semicircle *STR* was cut out from the cardboard. Given that the perimeter of the remainder of the cardboard is 72 cm, find the value of x and of y. PL4



11. Mr. Chee Hong instructed the students of Form 4 Kembara to draw a rectangular mural of length $7x$ m and width y m on the wall of the canteen. Two different shapes will be drawn on the wall as shown in the diagram on the right. *AED* is a semicircle. Given that the area of the wall is 28 m^2 and the perimeter of *ABCDE* is 26 m, find the diameter and radius of the semicircle. PL5



Exploring MATHEMATICS

Encik Awang, a chemist, has three types of solutions. Every day, he will prepare a few solutions of specific proportions. One day, Encik Awang intends to prepare a solution from three types of solutions. The first solution must contain 10% of acid, the second solution 40% of acid and the third solution 60% of acid. Encik Awang intends to prepare 1 000 litres of mixed solution with an acid concentration of 45%. The available amount of 40% acid solution is twice the amount of 10% acid solution. How much of each solution would you suggest Encik Awang use?

1. Write three equations based on the above statements.

2. Write your suggested workings to Encik Awang.

CHAPTER 4

Indices, Surds and Logarithms

What will be learnt?

- Laws of Indices
- Laws of Surds
- Laws of Logarithms
- Applications of Indices, Surds and Logarithms

List of Learning Standards

bit.ly/2VyB2Kk

KEYWORDS

Index	*Indeks*
Base	*Asas*
Rational number	*Nombor nisbah*
Irrational number	*Nombor tak nisbah*
Surd	*Surd*
Radical	*Radikal*
Recurring decimal	*Perpuluhan berulang*
Conjugate surd	*Surd konjugat*
Logarithm	*Logaritma*
Natural logarithm	*Logaritma jati*
Algebraic expression	*Ungkapan algebra*
Coefficient	*Pekali*

The population of the country needs to be estimated in order to plan the future of the nation. Knowledge on the growth rate of the Malaysian population will enable preparations to be made in various aspects, including medical facilities for children, new registrations for year 1 students and so on. In your opinion, how can the population of Malaysia in a particular year be estimated?

Did you Know?

John Napier was a famous Scottish mathematician who introduced logarithms. Logarithms is a mathematical tool which simplifies calculations, especially multiplications which are frequently utilised in astronomy.

For further information:

bit.ly/2pgYYWz

SIGNIFICANCE OF THIS CHAPTER

- The half-life of a radioactive substance is given by the function $N(t) = N_0e^{-\lambda t}$, where N_0 is the initial mass of the radioactive substance, $N(t)$ is the remaining mass of the radioactive substance post-decay, t is the time of decay and λ is the decay constant. By substituting the values of N_0, $N(t)$ and λ into the function, physicists can determine the time of decay of a radioactive substance.
- Biologists can determine the growth rate of bacteria from time to time if the bacteria were allowed to proliferate.
- The intensity of an earthquake can be determined by using exponential functions. This enables geoscientists to calculate its magnitude using the Richter scale.

Scan this QR code to watch a video on the population of Malaysia.

bit.ly/2qWcv70

4.1 Laws of Indices

INQUIRY 1 In pairs | 21st Century Learning

Aim: To recall the laws of indices

Instructions:

1. Scan the QR code or visit the link on the right.
2. Take note of the list of laws of indices. Cut out all shapes and paste at the corresponding laws of indices in the table.
3. Write an example of index law using algebraic expressions as shown in the following table.

Index law			Example
$a^m \times a^n$	=	a^{m+n}	$t^2 \times t^3 = t^{2+3} = t^5$

4. Display your partner's and your findings.
5. Along with your partner, move around and observe other teams' results.

bit.ly/2onDQy8

Simplifying algebraic expressions involving indices

You have learnt that a^n is an index number with a being the base and n being the index. How can an algebraic expression involving indices be simplified by using the laws of indices? Let's explore.

INQUIRY 2 Individual

Aim: To simplify algebraic expressions involving indices

Instructions:

1. List the laws of indices which you have learnt.
2. Scan the QR code or visit the link on the right.
3. Using the laws of indices you have listed down, simplify each of the provided algebraic expressions.
4. Click on the "Check Answer" button to check your answers.
5. Discuss with your friends the ways by which you have obtained your answers.

bit.ly/2MmURQD

From the results of Inquiry 2, we can deduce that:

> An algebraic expression involving index numbers can be simplified by using the laws of indices.

Example 1

Simplify the following algebraic expressions.

(a) $\dfrac{4^{2n} \times 4^{m}}{4^{n}}$

(b) $\dfrac{3^{m+2} - 3^{m}}{3^{m}}$

(c) $(5x^{-1})^3 \times 4xy^2 \div (xy)^{-4}$

(d) $4a^3b^2 \times (4ab^3)^{-4}$

Solution

(a) $\dfrac{4^{2n} \times 4^{m}}{4^{n}} = 4^{2n+m-n}$
$= 4^{n+m}$

(b) $\dfrac{3^{m+2} - 3^{m}}{3^{m}} = \dfrac{3^{m} \times 3^{2} - 3^{m}}{3^{m}}$
$= \dfrac{3^{m}(3^{2} - 1)}{3^{m}}$
$= 8$

(c) $(5x^{-1})^3 \times 4xy^2 \div (xy)^{-4}$
$= \dfrac{(5x^{-1})^3 \times 4xy^2}{(xy)^{-4}}$
$= 5^3x^{-3} \times 4xy^2 \times (xy)^4$
$= 125 \times 4 \times x^{-3+1+4} \times y^{2+4}$
$= 500x^2y^6$

(d) $4a^3b^2 \times (4ab^3)^{-4}$
$= 4a^3b^2 \times \dfrac{1}{(4ab^3)^4}$
$= \dfrac{4a^3b^2}{256a^4b^{12}}$
$= \dfrac{1}{64ab^{10}}$

Example 2

Simplify the following algebraic expressions.

(a) $a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}}$

(b) $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}}$

(c) $\sqrt[3]{a^2} \times \sqrt[2]{a^{-3}}$

(d) $a^{-\frac{1}{2}}(a^{\frac{3}{2}} + 2a^{\frac{1}{2}} - 3a^{-\frac{1}{2}})$

Solution

(a) $a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}} = 2 \times a^{-\frac{1}{3}} \times a^{-\frac{1}{2}}$
$= 2a^{-\frac{1}{3}+\left(-\frac{1}{2}\right)}$
$= 2a^{-\frac{5}{6}}$
$= \dfrac{2}{a^{\frac{5}{6}}}$

(b) $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}} = 2a^{-2} \div a^{-\frac{3}{2}}$
$= 2a^{-2-\left(-\frac{3}{2}\right)}$
$= 2a^{-\frac{1}{2}}$
$= \dfrac{2}{a^{\frac{1}{2}}}$

(c) $\sqrt[3]{a^2} \times \sqrt[2]{a^{-3}} = a^{\frac{2}{3}} \times a^{-\frac{3}{2}}$
$= a^{\frac{2}{3}+\left(-\frac{3}{2}\right)}$
$= a^{\frac{2}{3}-\frac{3}{2}}$
$= a^{-\frac{5}{6}}$
$= \dfrac{1}{a^{\frac{5}{6}}}$

(d) $a^{-\frac{1}{2}}(a^{\frac{3}{2}} + 2a^{\frac{1}{2}} - 3a^{-\frac{1}{2}})$
$= a^{-\frac{1}{2}} \times a^{\frac{3}{2}} + a^{-\frac{1}{2}} \times 2a^{\frac{1}{2}} - a^{-\frac{1}{2}} \times 3a^{-\frac{1}{2}}$
$= a^{-\frac{1}{2}+\frac{3}{2}} + 2a^{-\frac{1}{2}+\frac{1}{2}} - 3a^{-\frac{1}{2}-\frac{1}{2}}$
$= a^1 + 2a^0 - 3a^{-1}$
$= a + 2 - \dfrac{3}{a}$

Example 3

Show that

(a) $7^{2x-1} = \frac{49^x}{7}$

(b) $3^{x+4} + 3^{x+5} + 3^x$ is divisible by 25 for all positive integers x.

Solution

(a) $7^{2x-1} = \frac{7^{2x}}{7}$

$= \frac{49^x}{7}$

(b) $3^{x+4} + 3^{x+5} + 3^x = 3^x(3^4) + 3^x(3^5) + 3^x$

$= 3^x(81 + 243 + 1)$

$= 3^x(325)$

Since 325 is a multiple of 25, thus $3^{x+4} + 3^{x+5} + 3^x$ is divisible by 25 for all positive integers x.

Self Practice 4.1

1. Simplify the following algebraic expressions.

(a) $\frac{5^{3x} \times 5^x}{5^{-x}}$ (b) $\frac{7^{b-2} - 7^b}{7^{b+3}}$ (c) $\frac{9^{a-3} + 9^{a+4}}{81}$

(d) $c^4d^3 \times c^3d^5$ (e) $(xy^2)^3 \times x^3y^5$ (f) $(7x^{-1})^2 \times (49^{-2}xy)^3$

(g) $(3x^2y)^3 \times (x^3)^4 \div x^{16}y^2$ (h) $(p^2q^{-1})^5 \times q^8$ (i) $(pq^5)^4 \times p^3$

(j) $(49^{-2}xy)^3 \div (7xy)^{-2}$ (k) $20x^{-7}y^2 \div 4x^3y^{-4}$ (l) $6a^7b^{-2} \div 36a^3b^{-4}$

2. Simplify the following algebraic expressions.

(a) $a^{\frac{1}{3}} \times 2a^{-\frac{1}{2}}$ (b) $\frac{4a^3}{a^{-\frac{3}{5}}}$

(c) $\sqrt[5]{a^7} \times \sqrt[4]{a^{-9}}$ (d) $a^{-\frac{3}{2}}\left(a^{\frac{1}{2}} + 3a^{-\frac{3}{2}} - 3a^{-\frac{5}{2}}\right)$

3. Show that

(a) $4^{3a-2} = \frac{64^a}{16}$ (b) $9^{2a+2} = 81(81^a)$ (c) $7^{3a-4} = \frac{343^a}{2\,401}$

4. Show that $4^{x+2} + 4^{x+1} + 4^x$ is divisible 7 for all positive integers x.

Solving problems involving indices

Equations involving indices can be solved as follows:

If $a^m = a^n$, then $m = n$ or if $a^m = b^m$, then $a = b$ when $a > 0$ and $a \neq 1$.

4.1.1 4.1.2

Example 4

Solve each of the following equations.

(a) $32^x = \dfrac{1}{8^{x-1}}$

(b) $a^5 = 243$

(c) $27(81^{3x}) = 1$

Solution

(a) $32^x = \dfrac{1}{8^{x-1}}$

$2^{5x} = 2^{-3(x-1)}$ ← Express both sides of equations in a same base

$5x = -3x + 3$ ← Compare the indices

$8x = 3$

$x = \dfrac{3}{8}$

(b) $a^5 = 243$

$= 3^5$ ← Express in index form

$a = 3$ ← Compare the indices

(c) $27(81^{3x}) = 1$

$3^3(3^4)^{3x} = 3^0$ ← $3^0 = 1$

$3^{3+12x} = 3^0$ ← $a^m \times a^n = a^{m+n}$

$3 + 12x = 0$

$12x = -3$

$x = -\dfrac{3}{12}$

$= -\dfrac{1}{4}$

MATHEMATICS POCKET

- If $5^x = 5^4$, then $x = 4$.
- If $x^5 = 5^5$, then $x = 5$.

QUICK COUNT

Given $3^x = \dfrac{9}{3^{2x}}$, find the value of x.

1. Press [3] [$x^{\square}$] [ALPHA] [)] [▶] [ALPHA] [CALC].
2. Press [9] [$\frac{\square}{\square}$] [3] [$x^{\square}$] [2] [ALPHA] [)].
3. Press [SHIFT] [CALC].
4. Press [=] to obtain the value of x.

CHAPTER 4

Example 5 MATHEMATICS APPLICATION

Husna has RM1 000 000. She invests that sum of money into an investment institution that offers an annual return of 6%. Husna's investment amount after n years is calculated with the equation $J = p(1 + k)^n$ using p as the investment at the beginning of the year and k as the annual return rate. Determine Husna's investment amount after 20 years.

4.1.2

CHAPTER 4

Solution

1. Understanding the problem

- Initial investment, p is RM1 000 000
- Return rate, k is 6% per annum
- Investment formula, $J = p(1 + k)^n$
- $n = 20$
- Determine the total investment after 20 years

2. Planning a strategy

Substitute the values of k, p and n into the investment formula.

3. Implementing the strategy

$$\begin{aligned}J &= p(1 + k)^n \\ &= 1\,000\,000\left(1 + \frac{6}{100}\right)^{20} \\ &= 1\,000\,000(1 + 0.06)^{20} \\ &= 1\,000\,000(3.207135) \\ &= 3\,207\,135\end{aligned}$$

Thus, Husna's total investment is RM3 207 135.

4. Making a conclusion

When $J = 3\,207\,135$ and $k = 0.06$, then

$$\begin{aligned}3\,207\,135 &= 1\,000\,000(1 + 0.06)^n \\ 3.207135 &= (1.06)^n \\ n &= 20\end{aligned}$$

Thus, $n = 20$ years.

Self Practice 4.2

1. Solve the following equations:

(a) $4^{x-1} = 8^{x+3}$ (b) $3^{x+3} - 3^{x+2} = 2$ (c) $8^{x-3} = \frac{4^{2x}}{64}$

2. A ball was released at a height of h cm from the surface of the Earth. The ball will bounce with 90% of its initial height after the ball hits the surface of the Earth. The height of the ball after l bounces is given by $h = 10 \times (0.9)^l$. Determine the height of the ball, in cm,

(a) when the ball was released,

(b) after 10 bounces.

Intensive Practice 4.1

Scan the QR code or visit bit.ly/2IAEoaM for the quiz

1. Simplify each of the following:

(a) $\frac{y^3(3zx)^2}{9x^3}$ (b) $\frac{z^4yx^2}{zxy^2}$ (c) $[(xy)^5 \times 2xy^3]^2$

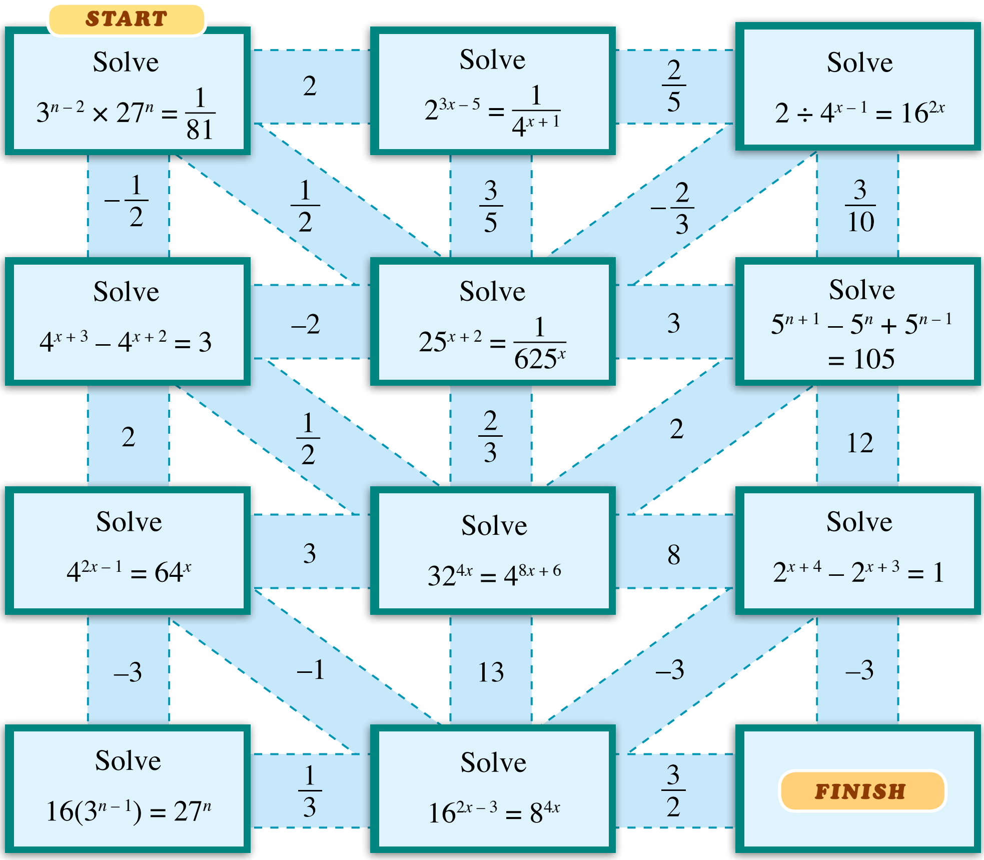
(d) $(ef^2)^3 \div (e^{-2}f^2)$ (e) $4.2x^4y^{14} \div 0.6x^9y^5$ (f) $(7x^{-1})^2 \times (49^{-2}xy)^3 \div (7xy)^{-2}$

2. If $2^{x-2} = 2(16)$, find the value of x.

3. Solve $25^x - 5^{3x-4} = 0$.

4. Solve $4(2^{m+1}) - 16^m = 0$.

5. Find the way to the FINISH box by choosing the correct answers.



6. In a research, a type of bacteria will multiply itself in 1 minute. The amount of bacteria at the start of the research was 300. The amount of bacteria after t minutes is given by $300(3^{t})$.
(a) Determine the amount of bacteria after 9 minutes.
(b) Determine the time, t, in minutes for the amount of the bacteria to be 72 900.

7. The population of country M can be estimated with the growth model, $P = A\left(1 + \frac{k}{100}\right)^{t}$ where P is the expected population, A is the population in year 2017, k is the growth rate and t is the number of years after 2017. The population of this country in 2017 was approximately 30 million. Assuming that this population increases with a rate of 3% each year, estimate the population of this country in the year 2050.

8. Mr. Prakesh invested RM20 000 in a bank with an annual interest rate of 10%. Mr. Prakesh's investment amount after t years can be determined by the formula $P = f(1 + r)^{t}$ where f is the initial investment value and r is the annual return rate. Determine Mr. Prakesh's investment amount after 10 years.

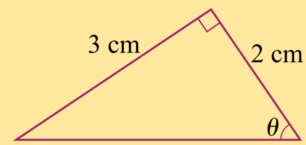
4.2 Laws of Surds

INQUIRY 3 In groups

Aim: To know surds

Instructions:

1. Observe the diagram on the right.
2. Without using a calculator, find the value of cosine θ and give the answer in the form of $\frac{a}{\sqrt{b}}$, where a and b are integers.
3. Discuss your group's findings.



We often face problems as mentioned above. How can problems involving surds be solved? Let's explore.

Comparing rational numbers and irrational numbers, and relating surds with irrational numbers

You have learnt about rational numbers, in other words, numbers that can be expressed in the form of $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Rational numbers can also be written in decimal form such as $\frac{1}{3} = 0.3333\ldots$ What is the connection between rational numbers and irrational numbers?

INQUIRY 4 In groups 21st Century Learning

Aim: To find the relation between surds and irrational numbers

Instructions:

1. Scan the QR code or visit the link on the right.
2. Browse the Internet to obtain information about surds.
3. Cut all the prepared number cards and paste them onto the table according to the correct classification as shown as the following.

bit.ly/315G1Ub

Rational numbers	Irrational numbers	
	Surd	Not surd
0.333333...		

4. Convert all decimals on the number cards to fractions. What can be concluded?
5. Each group will move to other groups to see their findings.
6. Discuss with team members about the findings of other groups.

From the results of Inquiry 4, it is found that:

(a) Decimals that can be coverted to fractions are **rational numbers**.
(b) Decimals that cannot be converted to fractions are **irrational numbers**.
(c) Numbers with radicals, if the values are integers or a recurring decimals, are **not surds**.

Surds are numbers in the square root form, that is $\sqrt{a}$, where a is any positive integer. Surds have infinite decimal places and are non-recurring. $\sqrt[n]{a}$ is called "surd a order n". For example, $\sqrt[3]{4}$ is called "surd 4 order 3". When a number cannot be simplified by eliminating the root, then that number is classified as a surd.

For instance,
(a) $\sqrt{2}$ cannot be simplified, therefore $\sqrt{2}$ is a surd.
(b) $\sqrt{4}$ can be simplified as 2, therefore $\sqrt{4}$ is not a surd.

Are all numbers with roots, surds? Observe the following table.

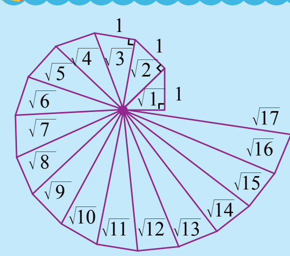
Number	Simplified number	Decimal	Surd or Not surd
$\sqrt{3}$	$\sqrt{3}$	1.7320508...	Surd
$\sqrt{\frac{1}{4}}$	$\frac{1}{2}$	0.5	Not surd
$\sqrt[3]{11}$	$\sqrt[3]{11}$	2.2239800...	Surd
$\sqrt[3]{27}$	3	3	Not surd
$\sqrt[5]{3}$	$\sqrt[5]{3}$	1.2457309...	Surd

From the table above, it is found that surds have non-recurring decimals. Therefore, surds are irrational numbers. Recurring decimals, such as, 54.565656… are sometimes written as $54.\dot{5}\dot{6}$ or $54.\overline{56}$.

MATHEMATICS POCKET

- Radical symbols are as follows.
  $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[5]{\ }$, $\sqrt[n]{\ }$
- Recurring decimals are decimals that can be converted to fractions. An example of a recurring decimal is 54.5656…

Mathematics Museum



In geometry, the first Theodorus spiral (also called square root spiral, Einstein spiral or Pythagorean spiral) was built by Theodorus from Cyrene. This spiral composed of right-angled triangles that were placed edge-to-edge.

4.2.1

Example 6

Convert the following recurring decimals to fractions.
(a) 0.676767…
(b) 12.645645645…

Solution

(a) Let,

$$N = 0.676767\ldots \quad \cdots ①$$
$$100N = 67.6767\ldots \quad \cdots ②$$
$$② - ①: 99N = 67$$
$$N = \frac{67}{99}$$

Multiply with a suitable integer so that the part with the recurring decimal can be eliminated.

Thus, $0.676767\ldots = \frac{67}{99}$.

(b) Let,

$$A = 12.645645645\ldots$$
$$A = 12 + N$$

Assume, $N = 0.645645645\ldots \quad \cdots ①$

$$1000N = 645.645645\ldots \quad \cdots ②$$
$$② - ①: 999N = 645$$
$$N = \frac{645}{999}$$
$$= \frac{215}{333}$$
$$A = 12 + \frac{215}{333}$$

Thus, $12.645645645\ldots = 12\frac{215}{333}$.

Mind Challenge

Convert the following fraction to a recurring decimal.

$$\frac{224}{495}$$

Smart TIPS

Is $\sqrt[n]{a} = n\sqrt{a}$?

$\sqrt{9} = (9)^{\frac{1}{2}} = 3$

$2\sqrt{9} = 2 \times 9^{\frac{1}{2}}$
$= 2 \times 3$
$= 6$

Since $3 \neq 6$, thus $\sqrt[n]{a} \neq n\sqrt{a}$.

Example 7

Determine whether the following terms are surds. Give your reason.

(a) $\sqrt[3]{125}$ (b) $\sqrt[5]{125}$ (c) $\sqrt[4]{\frac{16}{64}}$

Solution

Use a scientific calculator to obtain the values.

(a) $\sqrt[3]{125} = 125^{\frac{1}{3}}$
$= 5$

$\sqrt[3]{125}$ is not a surd because the value is an integer.

(b) $\sqrt[5]{125} = 2.6265278$

$\sqrt[5]{125}$ is a surd because it is a non-recurring decimal.

(c) $\sqrt[4]{\frac{16}{64}} = 0.7071067\ldots$

$\sqrt[4]{\frac{16}{64}}$ is a surd because it is a non-recurring decimal

Example 8

Is $\sqrt{4} = 2\sqrt{4}$? Explain.

Solution

$\sqrt{4} = 4^{\frac{1}{2}}$
$= 2$

,

$2\sqrt{4} = 2 \times 4^{\frac{1}{2}}$
$= 2 \times 2$
$= 4$

Since $2 \neq 4$, thus $\sqrt{4} \neq 2\sqrt{4}$. Generally, $\sqrt[n]{a} \neq n\sqrt{a}$.

Smart TIPS

$\sqrt[n]{a} \neq n\sqrt{a}$ because $\sqrt[n]{a} = a^{\frac{1}{n}}$ while $n\sqrt{a} = n \times a^{\frac{1}{2}}$.

Self Practice 4.3

1. Convert the following recurring decimals to fractions.
(a) 0.787878… (b) 3.57575757… (c) 0.345345345… (d) 13.567567567...

2. Determine whether the following terms are surds. Give your reason.
(a) $\sqrt[3]{127}$ (b) $\sqrt[4]{1\,125}$ (c) $\sqrt[6]{\frac{64}{729}}$ (d) $\sqrt[7]{\frac{79}{897}}$

Making and verifying conjectures on $\sqrt{a} \times \sqrt{b}$ and $\sqrt{a} \div \sqrt{b}$

INQUIRY 5 In groups

Aim: To verify conjectures on $\sqrt{a} \times \sqrt{b}$ and $\sqrt{a} \div \sqrt{b}$

Instructions:

1. Scan the QR code or visit the link on the right.
2. Click on the "Law 1" and "Law 2" boxes. Then, drag cursors *a* and *b*.
3. State the conjectures based on your observation regarding both laws.
4. With a scientific calculator, fill in the following tables by taking any positive integer *a* and *b*.

bit.ly/2MkBDv5

a	b	$(a \times b)$	$\sqrt{a}$	$\sqrt{b}$	Value of $\sqrt{a} \times \sqrt{b}$	$\sqrt{(a \times b)}$	Value of $\sqrt{(a \times b)}$
2	5	10	$\sqrt{2}$	$\sqrt{5}$	3.162...	$\sqrt{10}$	3.162...

4.2.1 4.2.2

a	b	$(a \div b)$	$\sqrt{a}$	$\sqrt{b}$	Value of $\sqrt{a} \div \sqrt{b}$	$\sqrt{(a \div b)}$	Value of $\sqrt{(a \div b)}$
10	5	2	$\sqrt{10}$	$\sqrt{5}$	1.414...	$\sqrt{2}$	1.414...

5. Compare the values of the 6th row and 8th row for both completed tables.

6. Were you able to verify the conjectures made? Discuss.

From the results of Inquiry 5, it is found that:

For $a > 0$ and $b > 0$,
(a) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (Law 1)
(b) $\sqrt{a} \div \sqrt{b} = \sqrt{\dfrac{a}{b}}$ (Law 2)

MATHEMATICS POCKET

If $a \geqslant 0$, then $\sqrt{a}$ is a real number and
$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = (a^2)^{\frac{1}{2}} = a$

Example 9

Write the following as single surds.

(a) $\sqrt{2} \times \sqrt{7}$

(b) $\dfrac{\sqrt{24}}{\sqrt{8}}$

(c) $\sqrt{3a} \times \sqrt{5a}$

(d) $\dfrac{\sqrt{21a}}{\sqrt{7a}}$

Solution

(a) $\sqrt{2} \times \sqrt{7} = \sqrt{2 \times 7}$
$= \sqrt{14}$

(b) $\dfrac{\sqrt{24}}{\sqrt{8}} = \sqrt{\dfrac{24}{8}}$
$= \sqrt{3}$

(c) $\sqrt{3a} \times \sqrt{5a} = \sqrt{3a \times 5a}$
$= \sqrt{15a^2}$
$= a\sqrt{15}$

(d) $\dfrac{\sqrt{21a}}{\sqrt{7a}} = \sqrt{\dfrac{21a}{7a}}$
$= \sqrt{3}$

Self Practice 4.4

1. Write the following as single surds.

(a) $\sqrt{2} \times \sqrt{3}$ (b) $\sqrt{3} \times \sqrt{5}$ (c) $\sqrt{3} \times \sqrt{3}$ (d) $\sqrt{5} \times \sqrt{6}$

(e) $\dfrac{\sqrt{8}}{\sqrt{3}}$ (f) $\dfrac{\sqrt{18}}{\sqrt{3}}$ (g) $\dfrac{\sqrt{20}}{\sqrt{5}}$ (h) $\dfrac{\sqrt{5} \times \sqrt{6}}{\sqrt{3}}$

Simplifying expressions involving surds

INQUIRY 6 Individual

Aim: To simplify expressions involving surds

Instructions:

1. Scan the QR code or visit the link on the right.
2. Drag the cursor to change the value of the surd.
3. Write down the surds that can be simplified and the surds that cannot be simplified.
4. Simplify $\sqrt{90}$ without using mathematical tools and technological devices.

bit.ly/2LQqH9u

From the results of Inquiry 6, it is found that $\sqrt{3}$ cannot be simplified but $\sqrt{9}$ can be simplified to 3. Besides, $\sqrt{90}$ can be written as $\sqrt{9 \times 10}$ or $\sqrt{9} \times \sqrt{10}$, therefore $\sqrt{90} = 3\sqrt{10}$.

Example 10

Write $\sqrt{18}$ in the form of $a\sqrt{b}$ where a and b are integers and a is the largest value.

Solution

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \times 2}\\ &= \sqrt{9} \times \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

← 9 is the largest perfect square number and a factor of 18

Self Practice 4.5

1. Mark (✓) on the correct statements.

$\sqrt{5}\sqrt{7}$ $= \sqrt{12}$ ☐	$3\sqrt{2} \times 2\sqrt{2}$ $= 6\sqrt{2}$ ☐	$\sqrt{260}$ $= 2\sqrt{65}$ ☐	$(\sqrt{16}\sqrt{36})^2$ $= 576$ ☐	$4\sqrt{7} \times 5\sqrt{7}$ $= 20\sqrt{21}$ ☐
$\dfrac{4\sqrt{8}}{2\sqrt{4}}$ $= 2\sqrt{2}$ ☐	$\dfrac{\sqrt{18}}{\sqrt{3}}$ $= \sqrt{15}$ ☐	$\dfrac{\sqrt{75}}{\sqrt{3}}$ $= 5$ ☐	$\dfrac{30\sqrt{27}}{6\sqrt{3}}$ $= 15$ ☐	$(\sqrt{81})^2$ $= 81$ ☐

4.2.2 4.2.3

CHAPTER 4

2. Write the following in the form of $a\sqrt{b}$ where a and b are integers and a is the largest value.

(a) $\sqrt{12}$ (b) $\sqrt{27}$ (c) $\sqrt{28}$ (d) $\sqrt{32}$

(e) $\sqrt{45}$ (f) $\sqrt{48}$ (g) $\sqrt{54}$ (h) $\sqrt{108}$

How to perform addition, subtraction and multiplication involving surds? Let's explore this in detail.

INQUIRY 7 In groups 21st Century Learning

Aim: To perform mathematical operations involving additions, subtractions and multiplications on surds

Instructions:

1. Scan the QR code or visit the link on the right.
2. Consider the expressions involving surds.
3. Click on the "Solution" button to see the calculation steps.
4. Click on "Other Questions" to see the next question.
5. Make a note about the calculation steps shown and explain to other classmates about your understanding on solving expressions involving surds.

bit.ly/31OY9To

From the results of Inquiry 7, it is found that:

Expressions involving surds can be simplified by performing addition, subtraction and multiplication operations of surds.

Example 11

Simplify the following expressions.

(a) $\sqrt{2} \times \sqrt{3} + \sqrt{6}$

(b) $\sqrt{7}(6 - \sqrt{7})$

(c) $\sqrt{18} - \sqrt{8}$

(d) $(6 + 2\sqrt{2})(1 + 3\sqrt{2})$

Solution

(a)
$$\begin{aligned}\sqrt{2} \times \sqrt{3} + \sqrt{6} &= \sqrt{2 \times 3} + \sqrt{6} \\ &= \sqrt{6} + \sqrt{6} \\ &= 2\sqrt{6}\end{aligned}$$

(b)
$$\begin{aligned}\sqrt{7}(6 - \sqrt{7}) &= 6\sqrt{7} - \sqrt{7} \times \sqrt{7} \\ &= 6\sqrt{7} - 7\end{aligned}$$

(c)
$$\begin{aligned}&\sqrt{18} - \sqrt{8} \\ &= \sqrt{9 \times 2} - \sqrt{4 \times 2} \\ &= \sqrt{9} \times \sqrt{2} - \sqrt{4} \times \sqrt{2} \\ &= 3\sqrt{2} - 2\sqrt{2} \\ &= (3 - 2)\sqrt{2} \\ &= \sqrt{2}\end{aligned}$$

(d)
$$\begin{aligned}&(6 + 2\sqrt{2})(1 + 3\sqrt{2}) \\ &= 6(1) + 6(3\sqrt{2}) + 2\sqrt{2}(1) + (2\sqrt{2})(3\sqrt{2}) \\ &= 6 + 18\sqrt{2} + 2\sqrt{2} + 12 \\ &= 18 + 20\sqrt{2}\end{aligned}$$

Example 12

Simplify each of the following in the form of $a\sqrt{b}$.

(a) $4\sqrt{27}$ (b) $7\sqrt{243}$ (c) $5\sqrt{75}$

Solution

(a) $4\sqrt{27} = 4\sqrt{9 \times 3}$
$= 4(3)\sqrt{3}$
$= 12\sqrt{3}$

(b) $7\sqrt{243} = 7\sqrt{81 \times 3}$
$= 7(9)\sqrt{3}$
$= 63\sqrt{3}$

(c) $5\sqrt{75} = 5\sqrt{25 \times 3}$
$= 5(5)\sqrt{3}$
$= 25\sqrt{3}$

In Example 12, notice that $12\sqrt{3}$, $63\sqrt{3}$ and $25\sqrt{3}$ have $\sqrt{3}$ as the irrational factor. Therefore, these three expressions are known as similar surds.

Numbers that do not contain a similar irrational factor are known as not similar surd. For example, the set of expressions $\sqrt{3}$, $2\sqrt[3]{3}$, $5\sqrt{6}$ and $7\sqrt[4]{3}$ are not similar surds.

Example 13

Determine whether the set of expressions $4\sqrt{12}$, $5\sqrt{18}$ and $5\sqrt{6}$ are similar surds or not similar surds.

Solution

$4\sqrt{12} = 4\sqrt{4 \times 3}$
$= 4(2)\sqrt{3}$
$= 8\sqrt{3}$

$5\sqrt{18} = 5\sqrt{9 \times 2}$
$= 5(3)\sqrt{2}$
$= 15\sqrt{2}$

$5\sqrt{6} = 5\sqrt{2 \times 3}$
$= 5\sqrt{6}$

The three expressions do not contain a similar irrational number factor. Thus, all of those expressions are not similar surds.

Self Practice 4.6

1. Simplify the following expressions involving surds.

(a) $3\sqrt{5} + 5\sqrt{5}$ (b) $7\sqrt{5} + 5\sqrt{5}$ (c) $7\sqrt{7} - 5\sqrt{7}$

(d) $\sqrt{6}(3\sqrt{6} - 5\sqrt{6})$ (e) $\sqrt{5}(4 + 5\sqrt{5})$ (f) $\sqrt{7}(3 - 5\sqrt{7})$

(g) $(4 + 5\sqrt{3})(3 + 5\sqrt{3})$ (h) $(7 - 5\sqrt{7})(3 + 5\sqrt{7})$ (i) $(9 + 5\sqrt{4})(3 - 5\sqrt{4})$

2. Determine whether the following sets of expressions are similar surds or not similar surds.

(a) $5\sqrt{80}$, $2\sqrt{58}$, $9\sqrt{45}$

(b) $3\sqrt{3}$, $4\sqrt{12}$, $5\sqrt{27}$

(c) $2\sqrt{125}$, $7\sqrt{5}$, $-7\sqrt{5}$

(d) $2\sqrt{12}$, $9\sqrt{24}$, $8\sqrt{5}$

(e) $3\sqrt{27}$, $-3\sqrt{27}$, $-\sqrt{3}$

4.2.3

Rationalising the denominators for expressions involving surds

Numbers that contain irrational denominators such as $\frac{1}{m\sqrt{a}}$, $\frac{1}{m\sqrt{a}+n\sqrt{b}}$ and $\frac{1}{m\sqrt{a}-n\sqrt{b}}$, where m and n are integers should be written by rationalising the denominators. The rules to rationalising the denominators are as follows:

(a) Multiply the numerator and denominator of $\frac{1}{m\sqrt{a}}$ with the conjugate surd $m\sqrt{a}$ so that the surd can be eliminated from the denominator.

(b) Multiply the numerator and denominator of $\frac{1}{m\sqrt{a}+n\sqrt{b}}$ with the conjugate surd $m\sqrt{a}-n\sqrt{b}$ so that the surd can be eliminated from the denominator.

(c) Multiply the numerator and denominator of $\frac{1}{m\sqrt{a}-n\sqrt{b}}$ with the conjugate surd $m\sqrt{a}+n\sqrt{b}$ so that the surd can be eliminated from the denominator.

CHAPTER 4

Example 14

Rationalise the denominator and simplify each of the following.

(a) $\frac{1}{5\sqrt{3}}$ (b) $\frac{1}{7\sqrt{2}+5\sqrt{3}}$ (c) $\frac{1}{2\sqrt{3}-5\sqrt{7}}$

Solution

(a) $\frac{1}{5\sqrt{3}} = \frac{1}{5\sqrt{3}} \times \frac{5\sqrt{3}}{5\sqrt{3}}$ ← Multiply with the conjugate surd

$$= \frac{5\sqrt{3}}{5 \times 5 \times \sqrt{3} \times \sqrt{3}}$$
$$= \frac{5\sqrt{3}}{75}$$
$$= \frac{\sqrt{3}}{15}$$

(b) $\frac{1}{7\sqrt{2}+5\sqrt{3}} = \frac{1}{7\sqrt{2}+5\sqrt{3}} \times \frac{7\sqrt{2}-5\sqrt{3}}{7\sqrt{2}-5\sqrt{3}}$ ← Multiply with the conjugate surd

$$= \frac{7\sqrt{2}-5\sqrt{3}}{(7\sqrt{2}+5\sqrt{3})(7\sqrt{2}-5\sqrt{3})}$$
$$= \frac{7\sqrt{2}-5\sqrt{3}}{(7\sqrt{2})^2-(5\sqrt{3})^2}$$
$$= \frac{7\sqrt{2}-5\sqrt{3}}{23}$$

MATHEMATICS POCKET

Rationalising using conjugate surds.

Surd	Conjugate surd
$m\sqrt{a}$	$m\sqrt{a}$
$m\sqrt{a}+n\sqrt{b}$	$m\sqrt{a}-n\sqrt{b}$
$m\sqrt{a}-n\sqrt{b}$	$m\sqrt{a}+n\sqrt{b}$

Smart TIPS

$\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = a$

$(a-\sqrt{b})(a+\sqrt{b}) = a^2 - b$

(c) $\dfrac{1}{2\sqrt{3}-5\sqrt{7}} = \dfrac{1}{2\sqrt{3}-5\sqrt{7}} \times \dfrac{2\sqrt{3}+5\sqrt{7}}{2\sqrt{3}+5\sqrt{7}}$ ← Multiply with the conjugate surd

$$= \frac{2\sqrt{3}+5\sqrt{7}}{(2\sqrt{3}-5\sqrt{7})(2\sqrt{3}+5\sqrt{7})}$$

$$= \frac{2\sqrt{3}+5\sqrt{7}}{(2\sqrt{3})^2-(5\sqrt{7})^2}$$

$$= -\frac{2\sqrt{3}+5\sqrt{7}}{163}$$

The conjugate surd for $2\sqrt{3}-5\sqrt{7}$ is $2\sqrt{3}+5\sqrt{7}$.

Example 15

Rationalise the denominator and simplify $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$.

Solution

$\dfrac{1+\sqrt{3}}{1-\sqrt{3}} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$ ← Multiply with the conjugate surd

$$= \frac{1+3+\sqrt{3}+\sqrt{3}}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

Mind Challenge

What is the conjugate surd of $1-\sqrt{3}$?

Example 16

Write $\dfrac{5+\sqrt{7}}{1+\sqrt{3}} + \dfrac{4-\sqrt{7}}{1-\sqrt{3}}$ as a single fraction.

Solution

$$\frac{5+\sqrt{7}}{1+\sqrt{3}} + \frac{4-\sqrt{7}}{1-\sqrt{3}} = \left(\frac{5+\sqrt{7}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}\right) + \left(\frac{4-\sqrt{7}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}\right)$$

$$= \frac{5-5\sqrt{3}+\sqrt{7}-\sqrt{21}+4+4\sqrt{3}-\sqrt{7}-\sqrt{21}}{(1+\sqrt{3})(1-\sqrt{3})}$$

$$= \frac{9-\sqrt{3}-2\sqrt{21}}{1-3}$$

$$= \frac{-9+\sqrt{3}+2\sqrt{21}}{2}$$

Smart TIPS

Multiply $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$ with a fraction in the form $\dfrac{c}{a+\sqrt{b}}$ to eliminate the surd from the denominator.

BRAINSTORMING

"The product of 2 irrational numbers is an irrational number."

Discuss and give your justification regarding this statement.

CHAPTER 4

4.2.4

Self Practice 4.7

1. Rationalise the denominator and simplify each of the following:

(a) $\dfrac{2}{\sqrt{5}}$ (b) $\dfrac{7}{\sqrt{2}}$ (c) $\dfrac{\sqrt{2}}{\sqrt{5}}$

(d) $\dfrac{\sqrt{3}}{2\sqrt{12}}$ (e) $\dfrac{1+\sqrt{3}}{\sqrt{12}}$ (f) $\dfrac{3+\sqrt{2}}{5-\sqrt{5}}$

(g) $\dfrac{6-\sqrt{3}}{9-\sqrt{12}}$ (h) $\dfrac{3+\sqrt{2}}{5-\sqrt{2}}+\dfrac{4-\sqrt{3}}{7+\sqrt{3}}$ (i) $\dfrac{7-\sqrt{5}}{5+\sqrt{5}}-\dfrac{6+\sqrt{3}}{6-\sqrt{3}}$

Solving problems involving surds

Example 17 MATHEMATICS APPLICATION

The diagram on the right shows a pyramid-shaped house. The triangle shape at the front of the house has an area of $(20\sqrt{3}-4)$ m² and the length of its base is $(4+4\sqrt{3})$ m. Determine the height of the triangle at the front of the house in the form of $(a+b\sqrt{3})$, where a and b are rational numbers.

Solution

1. Understanding the problem

- The area of the triangle = $(20\sqrt{3}-4)$ m²
- The length of the base of the triangle = $(4+4\sqrt{3})$ m
- Determine the height of the triangle in the form of $(a+b\sqrt{3})$

2. Planning a strategy

- Use the formula of the area of triangle = $\frac{1}{2}$ × base × height

3. Implementing the strategy

$$\frac{1}{2}\times(4+4\sqrt{3})\times t = 20\sqrt{3}-4$$
$$(2+2\sqrt{3})t = 20\sqrt{3}-4$$
$$t = \frac{20\sqrt{3}-4}{2+2\sqrt{3}}$$
$$= \frac{20\sqrt{3}-4}{2+2\sqrt{3}}\times\frac{2-2\sqrt{3}}{2-2\sqrt{3}}$$
$$= \frac{40\sqrt{3}-120-8+8\sqrt{3}}{-8}$$
$$= \frac{-128+48\sqrt{3}}{-8}$$
$$= 16-6\sqrt{3}$$

The height of the triangle part at the front of the house is $(16-6\sqrt{3})$ m.

4.2.4 4.2.5

4. Making a conclusion

$$\begin{aligned}\text{The area of triangle} &= \frac{1}{2} \times (4 + 4\sqrt{3}) \times (16 - 6\sqrt{3}) \\ &= (2 + 2\sqrt{3})(16 - 6\sqrt{3}) \\ &= 32 - 12\sqrt{3} + 32\sqrt{3} - 36 \\ &= (20\sqrt{3} - 4)\ \text{m}^2\end{aligned}$$

Example 18

Solve $x - 4\sqrt{x} + 3 = 0$.

Solution

$$x - 4\sqrt{x} + 3 = 0$$

$$(\sqrt{x} - 3)(\sqrt{x} - 1) = 0 \quad \leftarrow \text{Factorise}$$

$$\sqrt{x} - 3 = 0 \quad \text{or} \quad \sqrt{x} - 1 = 0$$

$$\sqrt{x} = 3 \qquad \sqrt{x} = 1$$

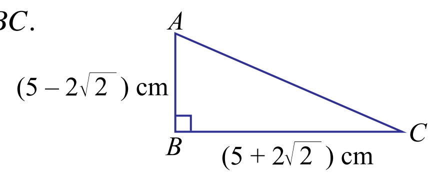
$$(\sqrt{x})^2 = 3^2 \qquad (\sqrt{x})^2 = 1^2$$

$$x = 9 \qquad x = 1$$

Self Practice 4.8

1. Triangle ABC has an angle of $ABC = 60°$, $AB = 3\sqrt{3}$ cm and $BC = 4\sqrt{3}$ cm. Determine the length of AC.

2. The diagram on the right shows a right-angled triangle ABC.
   (a) Determine the area of triangle ABC.
   (b) Determine the length of AC.



3. Solve the equation $2 + 3\sqrt{y} = 6\sqrt{3} + 5$. Write your answer in the form of $a + b\sqrt{3}$, where a and b are rational numbers.

4. Solve the following equations.
   (a) $\sqrt{2 - 7x} + 2x = 0$
   (b) $\sqrt{2x + 1} + \sqrt{2x - 1} = 2$
   (c) $\sqrt{4x + 3} - \sqrt{4x - 1} = 2$

4.2.5

Intensive Practice 4.2

Scan the QR code or visit bit.ly/33f32pd for the quiz

1. Write the following as single surds.

(a) $\sqrt{5} \times \sqrt{11}$ (b) $\sqrt{7} \times \sqrt{10}$ (c) $\dfrac{\sqrt{27}}{\sqrt{18}}$ (d) $\dfrac{\sqrt{48}}{\sqrt{8}}$

2. Write the following in the form of $a\sqrt{b}$, where a and b are integers and a is the largest value.

(a) $\sqrt{24}$ (b) $\sqrt{162}$ (c) $\dfrac{\sqrt{54}}{\sqrt{3}}$ (d) $\left(\dfrac{2\sqrt{6}}{3}\right)^2$

3. Simplify.

(a) $3\sqrt{10} + 5\sqrt{10}$ (b) $6\sqrt{11} - \sqrt{11}$ (c) $13\sqrt{13} - 2\sqrt{13}$

(d) $2\sqrt{45} + \sqrt{20}$ (e) $3\sqrt{27} - \sqrt{72}$ (f) $\sqrt{18} + \sqrt{27}$

(g) $3\sqrt{15} \times 7\sqrt{5}$ (h) $\sqrt{72} \times 4\sqrt{15}$ (i) $\sqrt{4}(2\sqrt{3}) - 5\sqrt{3}$

(j) $\sqrt{7}(3 + 7\sqrt{7})$ (k) $\sqrt{5}(7 - 5\sqrt{5})$ (l) $(3 + 3\sqrt{7})(3 + 5\sqrt{7})$

(m) $(7 + 5\sqrt{7})(3 - 5\sqrt{7})$ (n) $(7 - 5\sqrt{5})(3 - 5\sqrt{5})$ (o) $\dfrac{\sqrt{112}}{\sqrt{7}}$

(p) $\dfrac{\sqrt{12}}{\sqrt{108}}$ (q) $\dfrac{\sqrt{88}}{2\sqrt{11}}$ (r) $\dfrac{9\sqrt{20}}{3\sqrt{5}}$

4. Given $A = 3\sqrt{5} + 7\sqrt{3}$, $B = 2\sqrt{5} - 7\sqrt{7}$ and $C = 2\sqrt{3} - 9\sqrt{8}$. Simplify

(a) $A + B$ (b) $A - C$ (c) $3A + 2B$ (d) $3A + B - 2C$

5. Rationalise the denominators and simplify the following expressions.

(a) $\dfrac{2}{\sqrt{5}}$ (b) $\dfrac{4}{3 - \sqrt{5}}$ (c) $\dfrac{4}{3 - 3\sqrt{5}}$

(d) $\dfrac{5}{2\sqrt{3} - \sqrt{2}}$ (e) $\dfrac{4 + \sqrt{5}}{3 - \sqrt{5}}$ (f) $\dfrac{\sqrt{3} - \sqrt{7}}{\sqrt{3} + \sqrt{7}}$

6. Write the following as single fractions.

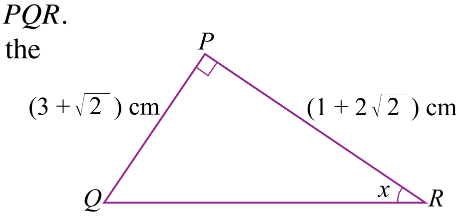
(a) $\dfrac{1}{1 + \sqrt{3}} + \dfrac{1}{1 - \sqrt{3}}$ (b) $\dfrac{2}{\sqrt{7} + \sqrt{2}} + \dfrac{1}{\sqrt{7} - \sqrt{2}}$ (c) $\dfrac{2}{4 - \sqrt{3}} + \dfrac{1}{4 + \sqrt{3}}$

7. The area of a rectangle is $(8 + \sqrt{10})$ cm². One of its sides has a length of $(\sqrt{5} + \sqrt{2})$ cm. Determine the length of the other right in the form of $a\sqrt{5} + b\sqrt{2}$.

8. The diagram on the right shows a right-angled triangle PQR.

(a) Determine the value of tan x. Write your answer in the form of $\dfrac{a + b\sqrt{2}}{c}$, where a, b and c are integers.

(b) Determine the area of triangle PQR. Write your answer in the form of $\dfrac{p + q\sqrt{2}}{r}$, where p, q and r are integers.



CHAPTER 4

4.3 Laws of Logarithms

Relating the equations in index form with logarithmic form and determining the logarithmic value of a number

An equation in index form can be written as $N = a^x$ where $a > 0$ and $a \neq 1$. N, a and x are variables. We can find the value of one variable if the value of the other two variables are given.
For example,
(a) if $81 = 9^x$, then $x = 2$
(b) if $1\,000 = a^3$, then $a = \sqrt[3]{1\,000}$
$= 10$
(c) if $N = 5^3$, then $N = 125$

Can you find the value of x of the following equations?
(a) $50 = 4^x$
(b) $69 = 7^x$
(c) $80 = 8^x$

What is the method that can be used? Let's explore in detail. Inquiry 8 will explain the methods to solve the equation above.

CHAPTER 4

INQUIRY 8 In groups

Aim: To relate the equations in index form and logarithmic form

Instructions:
1. Scan the QR code or visit the link on the right.
2. Click on the "Graph of equation in index form" box and observe the graph $f(x) = a^x$ that appears.
3. Then, click on the "Graph of equation in logarithmic form" box and observe the graph of $g(x) = \log_a(x)$ that appears.
4. Drag cursor a to the left and to the right. Write down your observations regarding the changes that happen to the graph when the value of a increases.
5. Drag cursor a to the value of 1. Does the graph $g(x) = \log_a x$ exist? What is the shape of the graph for $f(x) = a^x$ that was formed? Write down your findings.
6. Drag cursor a to negative values. Does the graph $f(x) = a^x$ and $g(x) = \log_a x$ exist? Write down your findings.
7. Discuss on the existence of logarithms for negative numbers and zero.
8. Then, verify if the following statements are true or false.
(a) $\log_a 1 = 0$ (b) $\log_a a = 1$

bit.ly/30Txh3t

4.3.1

From the results of Inquiry 8, it was found that the relationship between equations in index and logarithmic forms can be defined as follows:

$$\log_a N = x \Leftrightarrow N = a^x \text{ where } a > 0 \text{ and } a \neq 1$$

From the definition above, it can be concluded that:

$a^0 = 1 \Leftrightarrow \log_a 1 = 0$ and $a^1 = a \Leftrightarrow \log_a a = 1$

Thus, for any real numbers, $a > 0$ and $a \neq 1$, the following statement is true.

$$\log_a 1 = 0$$
$$\log_a a = 1$$

Notice that:

$$\log_a N \text{ is defined if } N > 0 \text{ and } a > 0, a \neq 1$$

For example, $\log_7 0$, $\log_{10} (-10)$, $\log_0 2$ and $\log_1 13$ are undefined.

The base of logarithms must be positive. Usually, 1 is not used as a base because $1^n = 1$ for any value of n.

If a common logarithm value is given for a number, that number can be determined with a scientific calculator. That number is called **antilogarithm** or **antilog** for short.

$$\text{If } \log_{10} N = x, \text{ then antilog } x = N$$

Based on the definition of logarithms of a number, we can convert an index equation to the logarithmic form.

The power of the index number is the logarithmic value

$$\text{Given } 16 = 2^4 \text{ thus } \log_2 16 = 4$$

The base of the index number is the base of the logarithm

Otherwise, we can convert an equation in logarithmic form to index form.

$$\text{If } \log_2 16 = 4, \text{ then } 16 = 2^4$$

Smart TIPS

$\log_a a^x = x$

MATHEMATICS POCKET

Index form	Logarithmic form
$4^0 = 1$	$\log_4 1 = 0$
$10^0 = 1$	$\log_{10} 1 = 0$
$7^1 = 7$	$\log_7 7 = 1$
$10^1 = 10$	$\log_{10} 10 = 1$

MATHEMATICS POCKET

The common logarithm is the logarithm in base 10. For example, $\log_{10} a = \lg a$

Mathematics Museum

The value of common logarithms can be determined by using a scientific calculator or a four-figure table.

Scan the QR code below to obtain a four-figure table.

bit.ly/2MvAcvb

INQUIRY 9 In pairs 21st Century Learning

Aim: To relate the graphs of exponential function and logarithmic function.

Instructions:

1. Copy and complete the table below for $y = 2^x$.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{8}$						

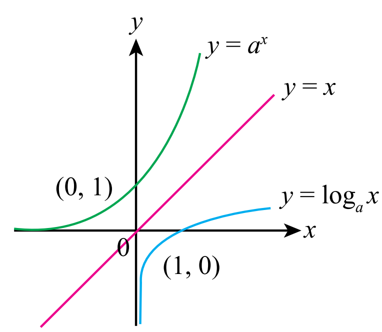
2. Then, copy and complete the table below for the inverse function $y = 2^x$, by exchanging the value of x and y.

x	$\frac{1}{8}$						
y	–3						

3. Draw the graph of y against x for $y = 2^x$ and its inverse function on the same axis.
4. Note down your observation about both of the graphs drawn.
5. Present your findings in front of the class.

From Inquiry 9, $f : x \to 2^x$, $x = f^{-1}(2^x)$.

Suppose $y = 2^x$,
thus $x = f^{-1}(y)$
$\log_2 y = \log_2 2^x$
$\log_2 y = x$
Substitute $x = \log_2 y$ into $x = f^{-1}(y)$
then, $f^{-1}(y) = \log_2 y$
or $f^{-1}(x) = \log_2 x$



Generally,

If $f : x \to a^x$, then $f^{-1} : x \to \log_a x$

Thus,

$y = \log_a x$ is the inverse of $a^y = x$

Example 19

Convert $2^4 = 16$ to logarithmic form.

Solution

$2^4 = 16$
$\log_2 16 = 4$

4.3.1

CHAPTER 4

Example 20

Convert $\log_3 27 = 3$ to index form.

Solution

$\log_3 27 = 3$

$3^3 = 27$

Example 21

Find the value of each of the following.

(a) $\log_{10} 7$ (b) $\log_{10} 79$ (c) $\log_{10} \left(\frac{3}{4}\right)^3$

Solution

(a) $\log_{10} 7 = 0.8451$

(b) $\log_{10} 79 = 1.8976$

(c) $\log_{10} \left(\frac{3}{4}\right)^3 = \log_{10} \left(\frac{27}{64}\right)$
$= -0.3748$

Example 22

Find the value of each of the following.

(a) $\log_5 625$ (b) $\log_6 7\,776$

Solution

(a) Let, $\log_5 625 = x$

$5^x = 625$

$5^x = 5^4$

$x = 4$

Thus, $\log_5 625 = 4$

(b) Let, $\log_6 7\,776 = y$

$6^y = 7\,776$

$6^y = 6^5$

$y = 5$

Thus, $\log_6 7\,776 = 5$

Example 23

(a) Determine the value of x if $\log_5 x = 3$.
(b) Determine the value of y if $\log_3 y = 4$.

Solution

(a) $\log_5 x = 3$

$x = 5^3$

$x = 125$

(b) $\log_3 y = 4$

$y = 3^4$

$y = 81$

Example 24

Determine the value of each of the following.

(a) antilog 0.1456 (b) antilog (–0.3976)

Solution

(a) antilog 0.1456 = 1.3983 (b) antilog (–0.3976) = 0.4003

Self Practice 4.9

1. Convert the following to logarithmic form.
(a) $3^4 = 81$ (b) $2^7 = 128$ (c) $5^3 = 125$ (d) $6^3 = 216$

2. Convert the following to index form.
(a) $\log_{10} 10\,000 = 4$ (b) $\log_{10} 0.0001 = -4$
(c) $\log_2 128 = 7$ (d) $\log_4 64 = 3$

3. Find the value of each of the following:
(a) $\log_{10} 9$ (b) $\log_{10} 99$
(c) $\log_{10} \left(\frac{5}{6}\right)^3$ (d) $\log_2 64$
(e) $\log_3 81$ (f) $\log_4 256$
(g) $\log_{10} 100\,000$

4. Solve the following equations.
(a) $\log_2 x = 5$ (b) $\log_8 x = 3$ (c) $\log_2 x = 8$

5. Determine the value of each of the following:
(a) antilog 2.1423 (b) antilog 1.3923
(c) antilog 3.7457 (d) antilog (–3.3923)
(e) antilog (–2.5676) (f) antilog (–4.5555)

Proving the laws of logarithm

INQUIRY 10 In groups 21st Century Learning

Aim: To prove the laws of logarithm

Instructions:

1. Scan the QR code or visit the link on the right.
2. Observe the examples of three laws of logarithm that is shown.
3. Drag cursors *a*, *b* and *n*. Observe the changes to the three laws of logarithm.
4. Discuss the three laws of logarithms and make a conclusion.
5. Make a short presentation about your findings.

bit.ly/2MiX9QP

4.3.1 4.3.2

From the results of Inquiry 10, three basic laws of logarithm are as follows:

If a, x and y are positive and $a \neq 1$, then

(a) $\log_a xy = \log_a x + \log_a y$ (Product law)

(b) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ (Division law)

(c) $\log_a x^n = n \log_a x$ for any real number n (Power law)

Each basic law of logarithm above can be proven as follows:
Let $x = a^p$ and $y = a^q$, then $p = \log_a x$ and $q = \log_a y$.

(a) $xy = a^p \times a^q = a^{p+q}$

Thus, $\log_a xy = p + q$ ← From the definition of logarithms

$\log_a xy = \log_a x + \log_a y$ ← Substitute $p = \log_a x$ and $q = \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^p}{a^q} = a^{p-q}$

Thus, $\log_a \dfrac{x}{y} = p - q$ ← From the definition of logarithms

$\log_a \dfrac{x}{y} = \log_a x - \log_a y$ ← Substitute $p = \log_a x$ and $q = \log_a y$

(c) $x^n = (a^p)^n = a^{pn}$

Thus, $\log_a x^n = pn$ ← From the definition of logarithms

$\log_a x^n = n \log_a x$ ← Substitute $p = \log_a x$

Example 25

Given $\log_5 15 = 1.6826$ and $\log_5 4 = 0.8614$. Determine the value of each of the following.

(a) $\log_5 60$ (b) $\log_5 12$ (c) $\log_5 100$

Solution

(a) $\log_5 60 = \log_5 (15 \times 4)$
$= \log_5 15 + \log_5 4$
$= 1.6826 + 0.8614$
$= 2.544$

(b) $\log_5 12 = \log_5 \left(\dfrac{60}{5}\right)$
$= \log_5 60 - \log_5 5$ ← $\log_a a^x = x$
$= 2.544 - 1$
$= 1.544$

(c) $\log_5 100 = \log_5 (25 \times 4)$
$= \log_5 25 + \log_5 4$
$= \log_5 5^2 + \log_5 4$
$= 2 \log_5 5 + 0.8614$
$= 2 + 0.8614$
$= 2.861$

Tech Whizz

Check your answers with the *Photomath* app. Scan the QR code below to download the *Photomath* app.

bit.ly/2Rg86YH

Example 26

Find the value of each of the following without using a calculator.

(a) $\log_5 750 - \log_5 6$

(b) $\log_3 8 + 2\log_3 6 - \log_3 \frac{96}{9}$

Solution

(a) $\log_5 750 - \log_5 6 = \log_5 \frac{750}{6}$
$= \log_5 125$
$= \log_5 5^3$
$= 3\log_5 5$ ← $\log_a a^x = x$
$= 3$

(b) $\log_3 8 + 2\log_3 6 - \log_3 \frac{96}{9} = \log_3 8 + \log_3 6^2 - \log_3 \frac{96}{9}$
$= \log_3 \left(8 \times 36 \div \frac{96}{9}\right)$
$= \log_3 27$
$= \log_3 3^3$
$= 3\log_3 3$ ← $\log_a a^x = x$
$= 3$

Self Practice 4.10

1. Given that $\log_7 4 = 0.712$ and $\log_7 5 = 0.827$. Evaluate each of the following:

(a) $\log_7 1\frac{1}{4}$ (b) $\log_7 28$ (c) $\log_7 100$ (d) $\log_7 0.25$

2. Evaluate each of the following without using a calculator.

(a) $\log_3 21 + \log_3 18 - \log_3 14$

(b) $2\log_4 2 - \frac{1}{2}\log_4 9 + \log_4 12$

(c) $\log_2 7 + \log_2 12 - \log_2 21$

Simplifying algebraic expressions using the laws of logarithms

Algebraic expressions involving logarithm can be simplified using the laws of logarithms.

Example 27

Express the following as single logarithms.

(a) $\log_a x + 3\log_a y$

(b) $2\log_a x - \frac{1}{2}\log_a y$

(c) $2\log_3 x + \log_3 y - 1$

Solution

(a) $\log_a x + 3\log_a y = \log_a x + \log_a y^3$
$= \log_a xy^3$

4.3.2 4.3.3

(b) $2 \log_a x - \frac{1}{2} \log_a y = \log_a x^2 - \log_a y^{\frac{1}{2}}$

$= \log_a \frac{x^2}{\sqrt{y}}$

(c) $2 \log_3 x + \log_3 y - 1 = \log_3 x^2 + \log_3 y - \log_3 3$

$= \log_3 \frac{x^2 y}{3}$

Example 28

If $p = \log_b 2$, $q = \log_b 3$ and $r = \log_b 5$, write the following in terms of p, q and/or r.

(a) $\log_b 6$
(b) $\log_b 45$
(c) $\log_b 0.2222\ldots$
(d) $\log_b \left(\frac{5\sqrt{3}}{2}\right)$

Solution

(a) $\log_b 6 = \log_b (2 \times 3)$
$= \log_b 2 + \log_b 3$
$= p + q$

(b) $\log_b 45 = \log_b (9 \times 5)$
$= \log_b 3^2 + \log_b 5$
$= 2 \log_b 3 + \log_b 5$
$= 2q + r$

(c) $\log_b 0.2222\ldots = \log_b \frac{2}{9}$
$= \log_b 2 - \log_b 9$
$= \log_b 2 - \log_b 3^2$
$= \log_b 2 - 2 \log_b 3$
$= p - 2q$

(d) $\log_b \left(\frac{5\sqrt{3}}{2}\right) = \log_b 5 + \log_b \sqrt{3} - \log_b 2$
$= \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2$
$= r + \frac{1}{2} q - p$

Mind Challenge

Can you determine the value of
(a) $\log_{10} (-6)$?
(b) $\log_{-10} 6$?

FLASHBACK

Suppose,
$A = 0.2222\ldots$ ①
$100A = 22.22\ldots$ ②
② − ① : $99A = 22$
$A = \frac{22}{99}$
$= \frac{2}{9}$

Self Practice 4.11

1. Write the following expressions as single logarithms.
   (a) $\log_2 x + \log_2 y^2$
   (b) $\log_b x - 3 \log_b y$
   (c) $\log_2 x + 3 \log_2 y$
   (d) $\frac{1}{2} \log_4 x + 2 - 3 \log_4 y$
   (e) $\log_3 m^4 + 2 \log_3 n - \log_3 m$

2. Given $\log_2 3 = p$ and $\log_2 5 = q$, express each of the following in terms of p and q.
   (a) $\log_2 10$
   (b) $\log_2 45$
   (c) $\log_2 \sqrt{15}$

Proving the relationship of $\log_a b = \frac{\log_c b}{\log_c a}$ and determining the logarithm of a number

If a, b and c are positive numbers, then, $a \neq 1$ and $c \neq 1$, then $\log_a b = \frac{\log_c b}{\log_c a}$

QR

Logarithmic base changes.

bit.ly/2NtoekS

The prove for the above statement are as follows:
Suppose $\log_a b = x$, then, $a^x = b$.

$$\log_c a^x = \log_c b$$ ← Use the same logarithmic base on both sides

$$x \log_c a = \log_c b$$ ← Power law of logarithm

$$x = \frac{\log_c b}{\log_c a}$$

Thus, $\log_a b = \frac{\log_c b}{\log_c a}$

Specifically:

If $b = c$, then $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$

By using the law of base change, any logarithmic base can be written and evaluated with base 10 or base e.

Logarithms with **base e** are known as **natural logarithms** and are written as $\log_e$, or ln. Base e is often used in mathematics, science and technology.

MATHEMATICS POCKET

ln a means $\log_e a$ with e as an exponent base. The number e is a non-recurring decimal, which is 2.7182…

Observe the following:
- $\log 10 = 1$
- $\ln e = 1$
- $\ln e^x = x$
- $e^{\ln x} = x$
- $10^{\log x} = x$

Example 29

Determine the following values by changing their bases to 10.
(a) $\log_{30} 4$ (b) $\log_2 0.45$

Solution

(a) $\log_{30} 4 = \frac{\log_{10} 4}{\log_{10} 30}$
$= \frac{0.6021}{1.4771}$
$= 0.408$

(b) $\log_2 0.45 = \frac{\log_{10} 0.45}{\log_{10} 2}$
$= \frac{-0.3468}{0.3010}$
$= -1.152$

Mind Challenge

Find the value of $\log_5 20$ using common logarithms and natural logarithms.

CHAPTER 4

4.3.4

Example 30

Convert each of the following to natural logarithms and evaluate them.

(a) $\log_6 254$ (b) $\log_{30} 4$

Solution

(a) $\log_6 254 = \dfrac{\log_e 254}{\log_e 6}$

$= \dfrac{\ln 254}{\ln 6}$

$= \dfrac{5.5373}{1.7918}$

$= 3.090$

(b) $\log_{30} 4 = \dfrac{\log_e 4}{\log_e 30}$

$= \dfrac{\ln 4}{\ln 30}$

$= \dfrac{1.3863}{3.4012}$

$= 0.408$

QUICK COUNT

Determine the solution to Example 30 with a scientific calculator.

1. Press [ln] [254] [)] [÷] [ln] [6] [)] [=].

2. The screen will show:

ln(254) ÷ ln(6)

3.090445097

Example 31

Given $\log_5 x = p$, express each of the following in terms of p.

(a) $\log_{25} x$ (b) $\log_x 25x^3$

Solution

(a) $\log_{25} x = \dfrac{\log_5 x}{\log_5 25}$

$= \dfrac{p}{2}$

(b) $\log_x 25x^3 = \dfrac{\log_5 25x^3}{\log_5 x}$

$= \dfrac{\log_5 5^2 + \log_5 x^3}{p}$

$= \dfrac{2 \log_5 5 + 3 \log_5 x}{p}$

$= \dfrac{2 + 3p}{p}$

Self Practice 4.12

1. Evaluate each of the following by converting it to base 10.

(a) $\log_3 22$ (b) $\log_6 1.32$ (c) $\log_5 18$ (d) $\log_4 0.815$

2. Convert each of the following to natural logarithms and evaluate them.

(a) $\log_7 225$ (b) $\log_9 324$ (c) $\log_{20} 379$

3. Given $\log_3 2 = t$, express each of the following in terms of t.

(a) $\log_2 9$ (b) $\log_9 8$ (c) $\log_2 18$ (d) $\log_2 \dfrac{9}{4}$

4. If $\log_2 m = a$ and $\log_2 n = b$, express each of the following in terms of a and b.

(a) $\log_4 m^2n^3$ (b) $\log_8 \dfrac{m}{n^2}$ (c) $\log_{mn} 8n$

Solving problems involving the laws of logarithms

Problems involving indices, such as $3^x = 70$ that cannot be expressed in the form of $a^x = a^y$ or $a^x = b^x$ can be solved by using logarithms.

Example 32

Solve the equation $3^{x-4} = 50^{x-3}$.

Solution

$$3^{x-4} = 50^{x-3}$$
$$(x-4)\log 3 = (x-3)\log 50 \quad \leftarrow \text{Use base 10 logarithm}$$
$$x\log 3 - 4\log 3 = x\log 50 - 3\log 50 \quad \leftarrow \log_{10} a = \log a$$
$$x\log 3 - x\log 50 = -3\log 50 + 4\log 3$$
$$x(\log 3 - \log 50) = -3\log 50 + 4\log 3$$
$$x = \frac{-3\log 50 + 4\log 3}{\log 3 - \log 50}$$
$$= 2.610$$

Example 33

Solve the following natural logarithmic equations.

(a) $\ln(4x-2) = 5$

(b) $10e^{2x} = 35$

Solution

(a)
$$\ln(4x-2) = 5$$
$$\log_e(4x-2) = 5$$
$$e^5 = 4x-2$$
$$148.4132 = 4x-2$$
$$4x = 150.4132$$
$$x = \frac{150.4132}{4}$$
$$= 37.603$$

(b)
$$10e^{2x} = 35$$
$$e^{2x} = 3.5$$
$$\ln e^{2x} = \ln 3.5$$
$$2x\ln e = \ln 3.5 \quad \leftarrow \ln e = 1$$
$$2x = \ln 3.5$$
$$x = \frac{\ln 3.5}{2}$$
$$= 0.626$$

Example 34 MATHEMATICS APPLICATION

The temperature of a block of steel rises from 30°C to T°C when it was heated for x seconds. Given $T = 30(1.2)^x$, determine

(a) the temperature of the steel when it is heated for 10.4 seconds,

(b) the time, x, in seconds, taken to increase the temperature of the block of steel from 30°C to 1 500°C.

CHAPTER 4

4.3.5

Solution

1. Understanding the problem

- Given the formula $T = 30(1.2)^x$
- The temperature rises from 30°C to T°C.
- Determine T when $x = 10.4$ seconds
- Determine x when the temperature of the block of steel rises from 30°C to 1 500°C.

2. Planning the strategy

- Substitute the value of x into the formula to find the value of T.
- Substitute the value of T into the formula to find the value of x.

3. Implementing the strategy

(a) $T = 30(1.2)^x$
$= 30(1.2)^{10.4}$
$= 199.8$°C

Thus, the temperature of the steel after 10.4 seconds is 199.8°C.

(b) $T = 30(1.2)^x$
$1\ 500 = 30(1.2)^x$
$\frac{1500}{30} = (1.2)^x$
$50 = (1.2)^x$
$\log 50 = x \log 1.2$
$x = \frac{\log 50}{\log 1.2}$
$= 21.4567$

Thus, the time taken for the block of steel to reach a temperature of 1 500°C is 21.4567 seconds.

4. Making a conclusion

(a) When $T = 199.8$°C, then
$199.8 = 30(1.2)^x$
$\frac{199.8}{30} = (1.2)^x$
$6.66 = (1.2)^x$
$\log 6.66 = x \log 1.2$
$x = \frac{\log 6.66}{\log 1.2}$
$= 10.4$ seconds

(b) When $x = 21.4567$ seconds, then
$T = 30(1.2)^{21.4567}$
$\approx 1\ 500$°C

Self Practice 4.13

1. Solve the following equations by giving answers in three decimal places.
(a) $4^{2x-1} = 7^x$ (b) $5^{2x-1} = 79^{x-1}$ (c) $7^{3x-1} = 50^x$

2. Solve the following equations by using natural logarithms. Give the answer in three decimal places.
(a) $\ln (5x + 2) = 15$ (b) $30e^{2x+3} = 145$ (c) $5e^{3x-4} = 35$
(d) $\ln (3x - 2) = 4$ (e) $41 - e^{2x} = 5$ (f) $\ln (x + 1)^2 = 4$

3. The price of a house after n years is given by RM260 000$\left(\frac{9}{8}\right)^n$. Determine the minimum number of years for the price of the house to exceed RM300 000 for the first time.

4. A company's savings after n years is RM2 000$(1 + 0.07)^n$. Determine the minimum number of years for their savings to exceed RM4 000.

5. After n years, Mr. Chong's money in a bank is RM4 000$(1.1)^n$. Calculate the number of years for Mr. Chong's money to exceed RM5100 for the first time.

6. The air pressure, in Hg, at a height of 10 km above sea level is given by $P = 760e^{-0.125h}$, where h is the height, in km, and $e = 2.718$. Determine the height above sea level if the pressure at that height is 380 mm Hg.

Intensive Practice 4.3

Scan the QR code or visit bit.ly/2osOqUo for the quiz

1. Given $\log_5 3 = 0.683$ and $\log_5 7 = 1.209$. Without using a calculator or four-figure tables, calculate $\log_5 1$ and $\log_7 75$.

2. Given $\log_a 3 = x$ and $\log_a 5 = y$, express $\log_a \left(\frac{45}{a^3}\right)$ in terms of x and y.

3. Determine the value of $\log_4 8 + \log_r \sqrt{r}$.

4. Without using a calculator or four-figure tables, simplify $\frac{\log_{12} 49 \times \log_{64} 12}{\log_{16} 7}$.

5. Given $\log_{10} x = 2$ and $\log_{10} y = -1$, prove that $xy - 100y^2 = 9$.

6. Given $\log_5 2 = m$ and $\log_5 7 = p$, express $\log_5 4.9$ in terms of m and p.

7. Simplify $\log_2 (2x + 1) - 5 \log_4 x^2 + 4 \log_2 x$.

8. Given that $\log_2 xy = 2 + 3 \log_2 x - \log_2 y$, express y in terms of x.

9. Given $\log_2 b = x$ and $\log_2 c = y$, express $\log_4 \left(\frac{8b}{c}\right)$ in terms of x and y.

10. The intensity of a sound, in decibel, is calculated by using the formula $d = 10 \log_{10} \left(\frac{P}{P_0}\right)$ where d is the intensity of sound, in decibel, P is the intensity of sound, in Watt and P_0 is the weakest intensity of sound that can be detected by the human ears, in Watt and it is a constant. In a house, a hot water pump has an intensity of sound of 50 decibels and a wattage of 10^{-7} Watts while a dishwasher has a sound intensity of 62 decibels.
    (a) Calculate the value of P_0.
    (b) Determine the wattage, in Watts, for the dishwasher and the hot water pump.
    (c) A wattage for sound that exceeds 100 Watts is said to be painful to the human ears. State the minimum intensity of sound, in decibel, that is considered to be painful to the human ears.

11. The population growth in a certain country is $P = 2\ 500\ 000e^{0.04t}$ where t is the number of years after year 2020 and $e = 2.718$.
    (a) What is that country's population in 2020?
    (b) What is that country's population in 2030?
    (c) In which year will that country's population exceed 50 000 000?

CHAPTER 4

4.3.5

4.4 Applications of Indices, Surds and Logarithms

Solving problems involving indices, surds and logarithms

Example 35 MATHEMATICS APPLICATION

An entomologist found that a grasshopper infestation towards plants spreads across an area of $A(n) = 1\,000 \times 2^{0.2n}$ acres, where n is the number of weeks after the initial observation.

(a) Determine the area of infestation at the beginning.

(b) Determine the area of infestation after

(i) 5 weeks, (ii) 10 weeks.

(c) How much time is needed for the infestation to spread across an area of 8 000 acres?

Solution

1. Understanding the problem

- Given $A(n) = 1\,000 \times 2^{0.2n}$
- $n = 0$, $n = 5$, $n = 10$
- $A = 8\,000$ acres

2. Planning the strategy

- Substitute the value of n into the given formula.
- Substitute the value of A into the given formula.

3. Implementing the strategy

(a) $A(n) = 1\,000 \times 2^{0.2n}$
$A(0) = 1\,000 \times 2^{0.2(0)}$
$= 1\,000 \times 1$
$= 1\,000$ acres

(b) (i) $A(n) = 1\,000 \times 2^{0.2n}$
$A(5) = 1\,000 \times 2^{0.2(5)}$
$= 1\,000 \times 2^1$
$= 2\,000$ acres

(ii) $A(n) = 1\,000 \times 2^{0.2n}$
$A(10) = 1\,000 \times 2^{0.2(10)}$
$= 1\,000 \times 2^2$
$= 4\,000$ acres

(c) $8\,000 = 1\,000 \times 2^{0.2n}$
$2^{0.2n} = 8$
$2^{0.2n} = 2^3$
$0.2n = 3$
$n = 15$

Thus, the time taken for the infestation to spread across an area of 8 000 acres is 15 weeks.

4. Making a conclusion

(a) When $A = 1\,000$,
$1\,000 = 1\,000 \times 2^{0.2n}$
$2^{0.2n} = 1$
$0.2n \log 2 = \log 1$
$n = \dfrac{\log 1}{0.2 \times \log 2}$
$n = 0$ weeks

(b) (i) When $A = 2\,000$,
$2\,000 = 1\,000 \times 2^{0.2n}$
$2^{0.2n} = 2$
$0.2n \log 2 = \log 2$
$n = \dfrac{\log 2}{0.2 \times \log 2}$
$n = 5$ weeks

(ii) When $A = 4\,000$,
$4\,000 = 1\,000 \times 2^{0.2n}$
$2^{0.2n} = 4$
$0.2n \log 2 = \log 4$
$n = \dfrac{\log 4}{0.2 \times \log 2}$
$n = 10$ weeks

(c) When $n = 15$,
$A = 1\,000 \times 2^{0.2(15)}$
$= 8\,000$ acres

Self Practice 4.14

1. A gardener observes a bug infestation towards plants in his garden. He finds out that the area of insect infestation towards his plants is $A = 1\ 000 \times 2^{0.7n}$ acres, where n is the amount of weeks after the first week of initial observation. How long will it take for the insects to infest an area of 5 000 hectares?

2. The electric current that flows in an electrical circuit for t seconds after its switch is turned off is $I = 32 \times 4^{-t}$ amp.
   (a) Calculate the current flow when the switch is off.
   (b) Calculate the current flow after
   (i) 1 second, (ii) 2 seconds.
   (c) How long will it take for the current to reach 0.5 amps?

Intensive Practice 4.4

Scan the QR code or visit bit.ly/311IqiH for the quiz

1. Mr. Ramasamy keeps RM1 000 in a bank. The amount of money rises by $W = 1\ 000(1.09)^t$ after t years. Calculate
   (a) the amount of money after 5 years,
   (b) the time taken, t, in years, for the money to rise from RM1 000 to RM1 200.

2. The remaining radioactive substance of uranium after t years is $W(t) = 50 \times 2^{-0.0002t}$ gram, where $t \geqslant 0$.
   (a) Determine the initial mass of the uranium.
   (b) Determine the time that is needed for the uranium to weigh 8 grams.

3. The mass, J of a bacteria after time t, in hours is $J = 25 \times e^{0.1t}$ gram.
   (a) Show that the time taken for the bacterial mass to reach 50 grams is 10 ln 2 hours.
   (b) Determine the time taken in two decimal places.

SUMMARY OF CHAPTER 4

INDICES, SURDS AND LOGARITHMS

Indices

Laws of Indices

$a^m \times a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{mn}$

Surds

- $\sqrt{a} \times \sqrt{a} = a$
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $(a + \sqrt{b}) \times (a - \sqrt{b}) = a^2 - b$

Logarithms

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
- $\log_a b^n = n \log_a b$
- $\log_a b = \dfrac{\log_c b}{\log_c a}$
- $\log_a b = \dfrac{1}{\log_b a}$

WRITE YOUR JOURNAL

Make a poster that contains all the laws of indices, surds and logarithms according to your creativity. Each stated law must be accompanied with an example of its usage. Then, hang your poster in the classroom.

MASTERY PRACTICE

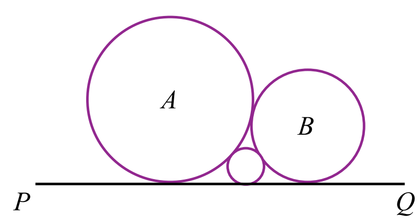
1. Solve the equation $4^{2x-1} + 4^{2x} = 4$. PL1

2. Solve the equation $5^{n+1} - 5^n + 5^{n-1} = 105$. PL2

3. If $\sqrt{5}x = \sqrt{3}x + \sqrt{7}$, find the value of x in the form of $\frac{\sqrt{a}}{b}$. PL2

4. If $\log_x a + \log_x \frac{1}{a} = t$, what is the possible value(s) of t? PL2

5. The diagram below shows three circles. Circle A has a radius of 2 cm and circle B has a radius of 1 cm.



PQ is a common tangent and all circles touch one another. Find the radius of the smallest circle. PL5

6. The temperature of a type of metal decreases from 100°C to T°C according to $T = 100(0.9)^x$ after x second. Calculate PL4
(a) the temperature of the metal after 5 seconds,
(b) the time taken, x, in seconds for the temperature of the metal to decrease from 100°C to 80°C.

7. After n years, the price of a car that was bought by Raju is RM60 000$\left(\frac{7}{8}\right)^n$. Determine the number of years for the price of the car to be below RM20 000 for the first time. PL4

8. Given $\log_x 3 = s$ and $\log_{\sqrt{y}} 9 = t$, express $\log_9 x^3y$ in terms of s and/or t. PL4

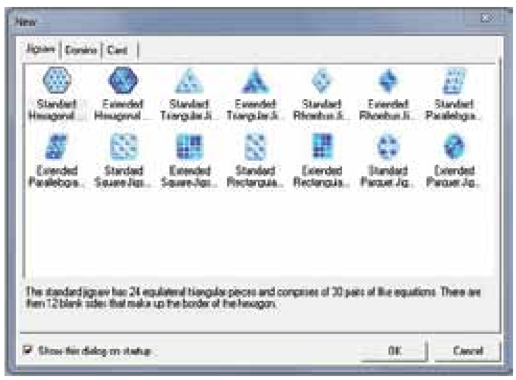
9. Two experiments were carried out to find the relationship between the variables x and y. Both experiments showed that the relationship between x and y is in accordance to $3(9^x) = 27^y$ and $\log_2 y = 2 + \log_2 (x - 2)$. Find the value of x and y that satisfy both experiments. PL5

10. The price of a car drops and can be determined with the equation $x \log_{10}\left(1 - \frac{2}{y}\right) = \log_{10} p - \log_{10} q$. In this equation, the car with y years of usage and price RMq will drop to RMp after being used for x years. A car is bought at RM100 000 has 20 years of usage. If the price of the car drops to RM10 000, find the years of usage for that car. PL5

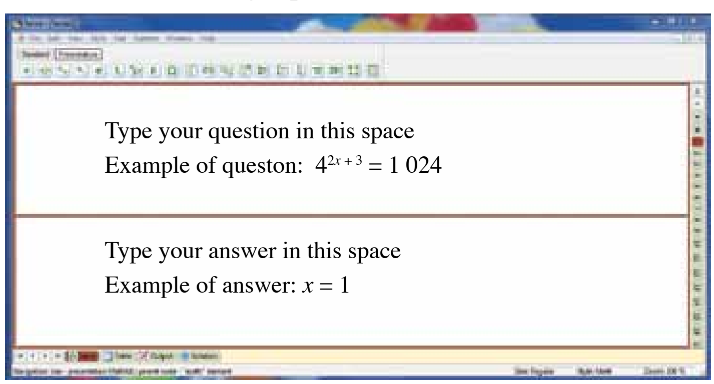
Exploring MATHEMATICS

Building an index and surd game with *Tarsia* software.

1. Download *Tarsia* software from *bit.ly/2SssDGz*.
2. Click on "*Standard Rhombus Jigsaw*" on the following window.



3. Type the question and answer in the relevant spaces. The number of questions that is need to be completed are shown on the right part of the screen.



4. Then, click on the "*Output*" button on the bottom part of the screen to generate the Jigsaw Puzzle. Print the Jigsaw Puzzle and cut it according to shape.
5. The Jigsaw Puzzle is ready to be used. Click on the "*Solution*" button to check the answers.

CHAPTER 4

CHAPTER 5

Progressions

What will be learnt?

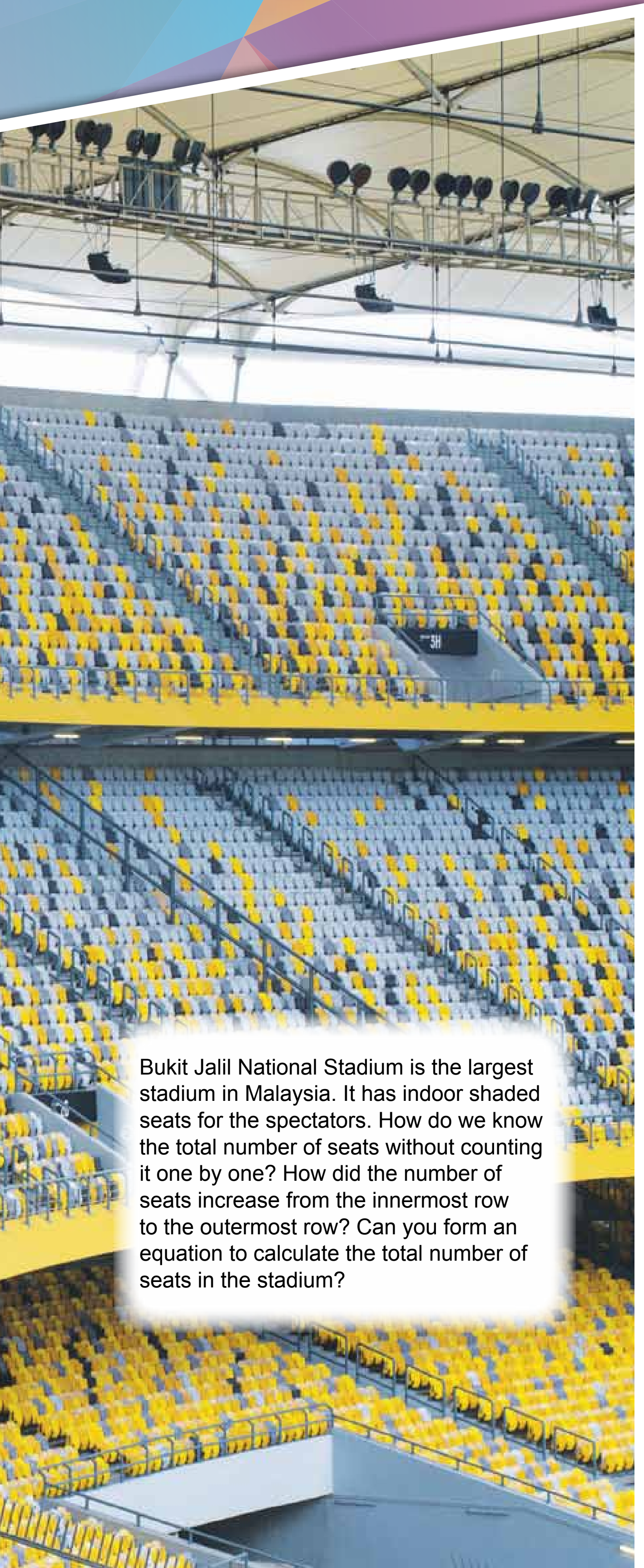
- Arithmetic Progressions
- Geometric Progressions

List of Learning Standards

bit.ly/2nGImr6

KEYWORDS

Sequence	*Jujukan*
Arithmetic progression	*Janjang aritmetik*
Common difference	*Beza sepunya*
Geometric progression	*Janjang geometri*
Common ratio	*Nisbah sepunya*
Sum to infinity	*Hasil tambah ketakterhinggaan*
Recurring decimal	*Perpuluhan berulang*



Bukit Jalil National Stadium is the largest stadium in Malaysia. It has indoor shaded seats for the spectators. How do we know the total number of seats without counting it one by one? How did the number of seats increase from the innermost row to the outermost row? Can you form an equation to calculate the total number of seats in the stadium?

Did you Know?

Carl Friedrich Gauss is a mathematician who is known as the Prince of Mathematics. His intelligence was proven since he was a child. Carl Friedrich Gauss corrected his father's wage calculation at 3 years old. At the age of 7, he was able to calculate the sum of 1 to 100 quickly and accurately.

For further information:

bit.ly/2p6owFX

SIGNIFICANCE OF THIS CHAPTER

The knowledge in solving progression problems is very important in the field of engineering, medicine, technology and economy. The knowledge in progression allows you to determine the total sums of large numbers with ease.

Scan this QR code to watch video of Bukit Jalil National Stadium.

bit.ly/2Vijima

5.1 Arithmetic Progressions

Identifying arithmetic progressions

Mr. Lee built stairs in his garden. He used eight bricks on the first level. For each subsequent level he increased another 8 bricks. The total number of bricks used on each level can be written in a progression of 8, 16, 24, … If Mr. Lee wanted to build 18 steps, how many bricks would be needed?

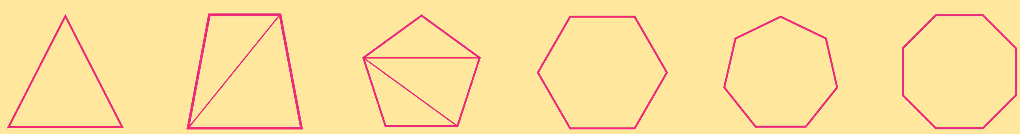
8, 16, 24, … is a finite sequence that follows a specific pattern. Sequence such as 3, -3, 3, -3, … is an infinite sequence. Each number in a sequence is known as terms, such that the first term is written as T_1, second term T_2 and so on until T_n, which is the n^{th} term.

INQUIRY 1 In groups

Aim: To understand arithmetic progressions

Instruction:

1. Observe the following polygons in which the number of sides of the consecutive polygons increase by one.



(a) (b) (c) (d) (e) (f)

2. Divide each polygon into triangular-shaped as shown in (b) and (c).
3. In the table, fill in the sum of interior angles for each of the given polygon.

Polygon arrangement, n	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Sum of interior angles	180°					

4. How do you obtain consecutive terms for the sum of interior angles in the polygons?
5. Explain the relationship between any two consecutive terms and state the finite value that relates the two terms.
6. Without plotting a graph, find the sum of interior angles for the tenth polygon arrangement.

From the results in Inquiry 1, it is given that the difference between any two consecutive terms of a sequence is a fixed constant. The constant is known as **common difference** and is represented by d. Thus:

$$d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$$

$$d \neq T_1 - T_2 \neq T_2 - T_3 \neq \ldots \neq T_{n-1} - T_n$$

Sequence that has common difference, d is known as **arithmetic progression**.

> Arithmetic progression is a sequence of numbers such that each term is obtained by adding a constant to the term before it.

Example 1

Determine whether each of the following sequence is an arithmetic progression. Give your justification.

(a) 358, 350, 342, …

(b) $\frac{2}{3}, 2, \frac{10}{3}, 5, \dots$

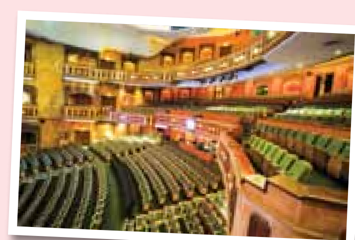
Solution

(a) $d_1 = 350 - 358 = -8$
$d_2 = 342 - 350 = -8$
This sequence is an arithmetic progression because $d_1 = d_2 = -8$.

(b) $d_1 = 2 - \frac{2}{3} = \frac{4}{3}$
$d_2 = \frac{10}{3} - 2 = \frac{4}{3}$
$d_3 = 5 - \frac{10}{3} = \frac{5}{3}$
This sequence is not an arithmetic progression because $d_1 = d_2 \neq d_3$.

Example 2

An auditorium has 15 chairs in the first row, 19 chairs in the second row, 23 chairs in the third row and so on. Determine whether the arrangement of chairs in each row follows an arithmetic progression. Give your justification.



Solution

Sequence: 15, 19, 23, …
$d_1 = 19 - 15 = 4$
$d_2 = 23 - 19 = 4$

Since, the difference of the progression is a fixed constant, which is 4. Thus, the arrangement of chairs in each row in the auditorium follows an arithmetic progression.

Self Practice 5.1

1. Find the common difference for each of the following arithmetic progression and state the method to obtain the arithmetic progression.

(a) $-35, -21, -7, \dots$

(b) $2\sqrt{3}, 5\sqrt{3}, 8\sqrt{3}, \dots$

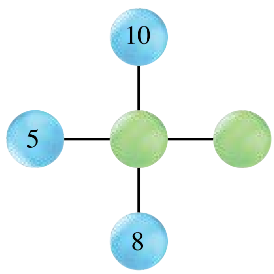
(c) $p + q, 2p, 3p - q, \dots$

(d) $\log_a 2, \log_a 2^4, \log_a 2^7, \dots$

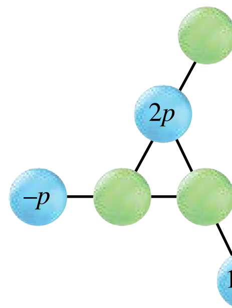
5.1.1

2. Determine whether each of the following sequence is an arithmetic progression and give your justification.
   (a) $9, 13, 17, 21, \ldots$
   (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$
   (c) $0.1, 0.01, 0.001, \ldots$
   (d) $5 - x, 5, 5 + x, \ldots$
3. Complete the network diagram below, given that the relationship of each of the following network is a consecutive term in an arithmetic progression.

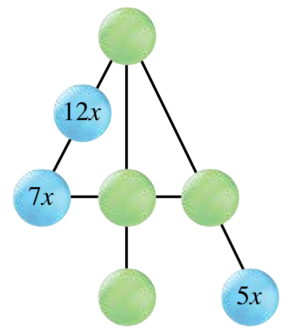
(a)



(b)



(c)



4. Azrul and Jonathan are placing national flags along the school corridor from the canteen to the staffroom. The distance between the first and second flag is 5m. The third flag is placed 10 m from the first flag and this is the pattern of arrangement for the rest of the flags until the end. Determine whether the arrangement of these flags follows arithmetic progression. Give justification for your answer.

Deriving the formula of the n^{th} term, T_n of arithmetic progression

INQUIRY 2 In groups

Aim: To derive the formula of the n^{th} term, T_n of arithmetic progression
Instruction:
1. Consider an arithmetic progression 2, 5, 8, 11, 14, ... Use the pattern of this sequence to complete the table.
2. Assume the first term of an arithmetic progression is a and the common difference is d.
3. Complete the table below.

Term	Value of term	Method to obtain the value of term	Formula (deduction method)
T_1	a	Does not have d	$T_1 = a + 0d$
T_2	$a + d$	Add d at T_1 term	$T_2 = a + 1d$
T_3	$a + d + d$	Add d at T_2 term	$T_3 = a + 2d$
⋮	⋮	⋮	⋮
T_n			

4. In your group, answer the following questions.
   (a) Express T_{20}, in terms of a and d.
   (b) State the relationship between the term T_n and its common difference.
   (c) Write a general formula for T_n.

From the results of Inquiry 2, the n^{th} term of an arithmetic progression can be written as:

$$T_n = a + (n - 1)d$$

Such that a is the first term, d is the common difference and n is the number of terms.

Example 3

(a) Find the 15^{th} term for the arithmetic progression $-4, 2, 8, \ldots$
(b) Find the 24^{th} term for the arithmetic progression $-6, 5, 16, \ldots$

Solution

(a) First term, $a = -4$
Common difference, $d = 2 - (-4) = 6$
15^{th} term, $T_{15} = -4 + (15 - 1)6$
$= 80$

(b) First term, $a = -6$
Common difference, $d = 5 - (-6) = 11$
24^{th} term, $T_{24} = -6 + (24 - 1)11$
$= 247$

QUICK COUNT

Based on Example 3, we can use scientific calculator to obtain the 15^{th} term.

1. Press [–4] [+] [(] [ALPHA] [)] [–] [1] [)] [(] [6] [)] [CALC]
   The screen displays:
   –4 + (x – 1)(6)
   x =
2. Press [15] [=]
   The screen displays:
   –4 + (x – 1)(6)
   80
3. Press [=] to enter the value of other terms.

Example 4

Given that an arithmetic progression with the first term is -6, the common difference is 11 and the n^{th} term is 126, find the value of n.

Solution

$a = -6, d = 11, T_n = 126$
$T_n = a + (n - 1)d$
$126 = -6 + (n - 1)(11)$
$126 = 11n - 17$
$n = 13$

Example 5

In a book fair, Siti wants to arrange books at the front section. She arranges the books by stacking them with the first book at 2 cm thickness at the bottom. However, the subsequent books have the same thickness, which is 1.5 cm. Find
(a) the total thickness of the books if Siti arranged 16 books,
(b) the number of books if the height of the books is 30.5 cm.

CHAPTER 5

5.1.2

Solution

(a) The sequence of the thickness of books: 2, 3.5, 5, 6.5, …

$a = 2, d = 1.5$

Total thickness of books at the 16th book. $= 2 + (16 - 1)(1.5)$

$= 24.5$ cm

Thus, the total thickness of 16 books is 24.5 cm.

(b) $T_n = 30.5$

$30.5 = 2 + (n - 1)(1.5)$

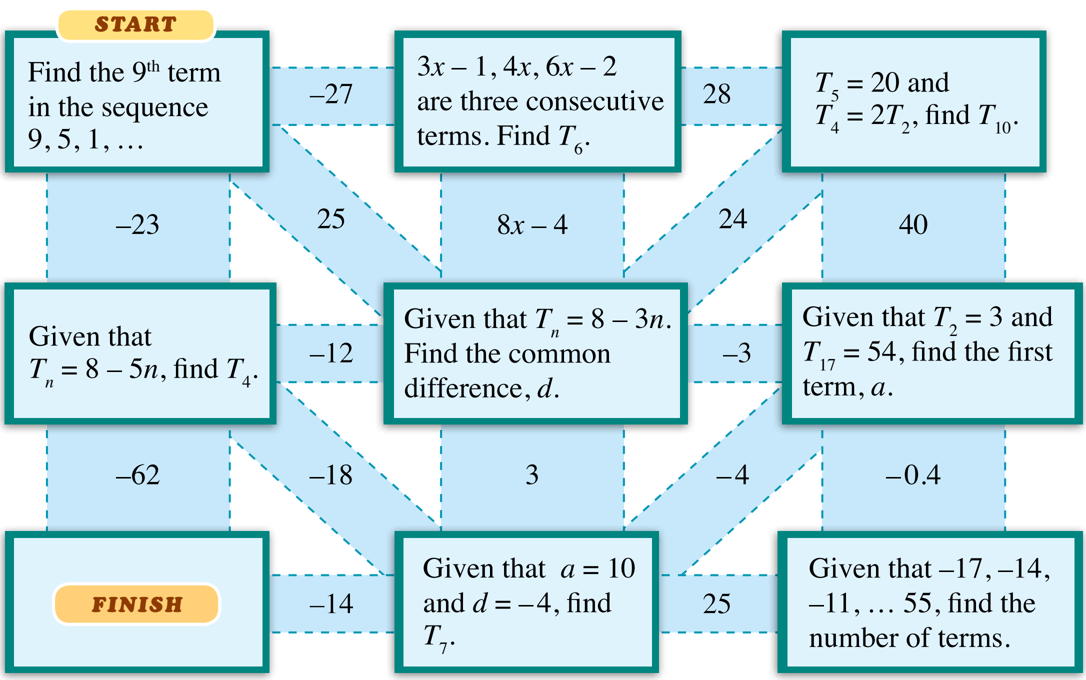
$n - 1 = 19$

$n = 20$

Thus, there are 20 books.

Self Practice 5.2

1. Find the way to the "FINISH" box by choosing the correct answer.



2. Encik Muiz starts to work in a company at a certain month. His first annual income is RM36 000 and the increment for the following year is RM1 000. Calculate

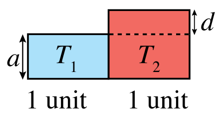
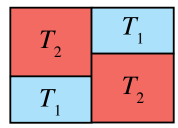
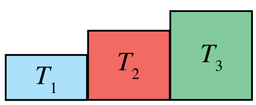
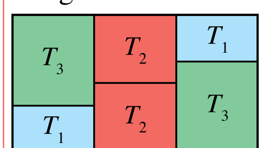
(a) the number of years in which Encik Muiz needs to work in order for him to receive twice his first annual income,

(b) the yearly increment if his salary during the 6th year is RM43 500.

Deriving the formula of sum of the first n terms, S_n, of the arithmetic progression

INQUIRY 3 In groups

Aim: To derive the formula of sum of the first n terms, S_n, of the arithmetic progression
Instruction:
1. Observe the table below.

<table>
<tr><th>Sum of terms</th><th>Number of grids based on the number of terms</th><th>Formula of a rectangle by deduction method</th></tr>
<tr><td>S_2</td><td>Diagram I

$T_1 = a$
$T_2 = a + (2 - 1)d$
$= a + d$</td><td>Diagram II

Area of a rectangle
$= (T_1 + T_2)2$
$= [a + a + (2 - 1)d]2$
$S_2 = \frac{2[2a + (2 - 1)d]}{2}$</td></tr>
<tr><td>S_3</td><td>Diagram III

$T_1 = a$
$T_2 = a + (2 - 1)d$
$T_3 = a + (3 - 1)d$</td><td>Diagram IV

Area of a rectangle
$= (T_1 + T_3)3$
$S_3 = \frac{(T_1 + T_3)3}{2}$
$= \frac{3}{2}[a + a + (3 - 1)d]$</td></tr>
<tr><td>S_4</td><td></td><td></td></tr>
<tr><td>⋮</td><td>⋮</td><td>⋮</td></tr>
<tr><td>S_n</td><td></td><td></td></tr>
</table>

2. Diagram I shows two grids with 1 unit width arranged side by side.
   - The height of blue grid is a unit which is represented by the first term T_1.
   - The height of red grid is d unit longer than that of the blue grid that represent the second term, $T_2 = a + d$ or $T_2 = a + (2 - 1)d$.
3. In Diagram II, the red grid is placed on top of the blue grid so that the total height is $T_1 + T_2 = a + a + (2 - 1)d$ unit. The blue grid is placed on top of the red grid so that the height becomes $T_1 + T_2 = a + a + (2 - 1)d$ unit.
4. Observe that both the blue and red grids become a rectangle. The sum of the blue grid and red grid, S_2, is half of the area formed. The sum can be written as $\frac{2[2a + (2 - 1)d]}{2}$.
5. Repeat step 1 to 3 to find S_4 and S_n.
6. Derive the formula of sum of the first n terms, S_n.

From the results of Inquiry 3, we found that the sum of the first n terms, S_n, of the arithmetic progression is derived by using the area of rectangle formed from the terms of the arithmetic progression.

Therefore, the sum of the first n terms, S_n, can be written as:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Such that a is the first term, n is the number of terms and d is the common difference.

5.1.3

Since $T_n = a + (n - 1)d$ is also a last term, l, hence the sum of the n^{th} term, S_n can be expressed as follow:

$$S_n = \frac{n}{2}[a + T_n] \quad \text{or} \quad S_n = \frac{n}{2}[a + l]$$

The n^{th} term of an arithmetic progression can be obtained by the formula of sum of the first n terms, S_n. For example, to find the value of 10^{th} term in an arithmetic progression, the sum of the first ten terms minus the sum of the first nine terms, which is $T_{10} = S_{10} - S_9$. In general:

$$T_n = S_n - S_{n-1}$$

Example 6

Given that an arithmetic progression 4, 7, 10, …, find
(a) the sum of the first 35 terms,
(b) the sum of the first n^{th} terms.

Solution

(a) First term, $a = 4$
Common difference, $d = 7 - 4 = 3$

$S_{35} = T_1 + T_2 + T_3 + \dots + T_{35}$

$S_{35} = \frac{35}{2}[2(4) + (35 - 1)(3)]$

$= 1\ 925$

(b) $S_n = \frac{n}{2}[2(4) + (n - 1)(3)]$

$= \frac{n}{2}[5 + 3n]$

Example 7

The sum of the first ten terms of an arithmetic progression is 230 and the sum of the subsequent ten terms is 630. Find the first term, a and the common difference, d for this arithmetic progression.

Solution

$S_{10} = \frac{10}{2}[2a + (10 - 1)d]$

$230 = 5(2a + 9d)$

$46 = 2a + 9d \quad \dots ①$

$S_{20} = \frac{20}{2}[2a + (20 - 1)d]$

$230 + 630 = 10(2a + 19d)$

$860 = 10(2a + 19d)$

$86 = 2a + 19d \quad \dots ②$

② – ①: $40 = 10d$

$d = 4$

Mind Challenge

In Example 7, why is $S_{20} = 230 + 630$? Explain your answer.

BRAINSTORMING

Discuss with your friends and prove that:
(a) $S_8 - S_5 = T_6 + T_7 + T_8$.
(b) $S_n - S_{n-1} = T_n$.

CHAPTER 5

5.1.3

Substitute $d = 4$ into ①,

$$46 = 2a + 9(4)$$
$$2a = 10$$
$$a = 5$$

Thus, the first term, a is 5 and the common difference, d is 4.

Example 8

A swarm of bees started to make a new beehive. 2 hexagonal holes were made on the first day. 5 hexagonal holes were made on the second day and 8 hexagonal holes were made on the third day and followed on until the beehive is ready.

Calculate

(a) the number of hexagonal holes on the 12th day,

(b) the minimum number of days if there are more than 1 000 hexagonal holes made.

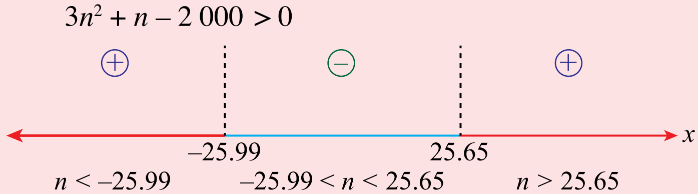
Solution

(a) The sequence of the number of hexagonal holes: 2, 5, 8, …
This sequence is an arithmetic progression.
Common difference, $a = 2$
First term, $d = 5 - 2 = 3$
Total number of hexagonal holes on the 12th day,

$$S_{12} = \frac{12}{2}[2(2) + (12 - 1)(3)]$$
$$= 222$$

(b) Total number of days, $S_n = T_1 + T_2 + T_3 + \dots T_n$

$$S_n > 1\ 000$$
$$\frac{n}{2}[2a + (n - 1)d] > 1\ 000$$
$$\frac{n}{2}[2(2) + (n - 1)(3)] > 1\ 000$$
$$n[1 + 3n] > 2\ 000$$
$$3n^2 + n > 2\ 000$$
$$3n^2 + n - 2\ 000 > 0$$



$n > \dfrac{153.92}{6}$ or $n < -\dfrac{155.92}{6}$

$n > 25.65$ $\quad < -25.99$ (Ignore)

Thus, the minimum number of days to make more than 1 000 hexagonal holes is 26 days.

Smart TIPS

Arithmetic progression is written in the form of $T_1, T_2, T_3, \dots$ whereas arithmetic series is written in the form of $T_1 + T_2 + T_3 + \dots$

MATHEMATICS POCKET

The beehive is made of a combination of hexagonal prismatic cells so that there are no space formed in between the hexagonal shapes. Thus, bees do not need to use a lot of wax to make their hives. The surface area of the hexagonal shapes is the largest compared to other shapes.

Scan the QR code to check out the reason beehive is hexagonal.

bit.ly/2AYE4hM

FLASHBACK

If $3n^2 + n - 2\ 000 = 0$, then

$n = \dfrac{-1 \pm \sqrt{1^2 - 4(3)(-2\ 000)}}{2(3)}$,

and $n = 25.65$ or $n = -25.99$

Mind Challenge

In Example 8, why is the value –25.99 not taken into account?

Self Practice 5.3

1. Find the sum of the following arithmetic progression.

(a) $-20, -15, -10, \ldots, 100$ (b) $\frac{3}{5}, \frac{6}{5}, \frac{9}{5}, \ldots$ till the first 23 terms.

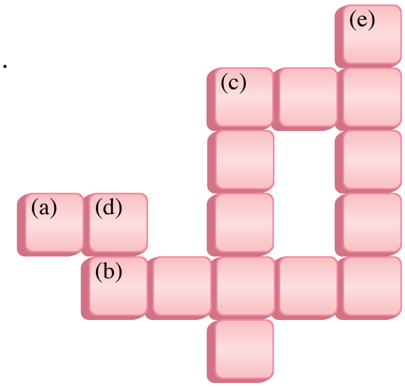
2. Complete the crossword puzzle.

Horizontal:

(a) Find the sum of the arithmetic series $38 + 34 + 30 + \ldots$ till the first 18 terms.

(b) Find the sum of the first 100 terms of an arithmetic progression such that the first term is -10 and the common difference is 6.

(c) Find the first term of the arithmetic progression in which the sum of the first 42 terms is 5 838 and the last term is -22.



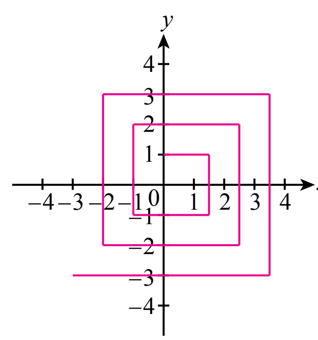
Vertical:

(c) Calculate S_{140} of an arithmetic progression that has 140 terms in which the first term and the last term are 2 and 449 respectively.

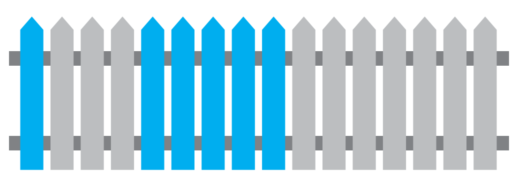
(d) Calculate the value of n of an arithmetic progression as such that the first term is -15, the common difference is -3 and the sum of the first n terms is $-1\ 023$.

(e) Calculate the sum of 200 terms after the first 50 terms of an arithmetic progression in which the sum of the first n terms is $S_n = \frac{n}{2}[n + 1]$.

3. The diagram on the right shows the pattern drawn on a Cartesian plane. The final line on the plan is parallel to the y-axis and passes through $x = -10$. Find the sum of the length of the overall pattern.



4. The diagram on the right shows a fence that is made of wood. The fence is painted with alternate blue and grey as shown in the diagram. The number of wood pieces painted with the same colour increases with the rate as shown in the diagram. If there are only 200 wood pieces,



(a) calculate the number of wood pieces that can be painted with the same colour. Hence, find the number of remaining wood pieces, if any.

(b) state the colour of the last wood piece and then, calculate the number of the wood pieces used for that particular colour.

Solving problems involving arithmetic progressions

Example 9 MATHEMATICS APPLICATION

Encik Suhaimi is a chicken breeder that has 1 500 chickens. He plans to sell 200 chickens each day. He gives food to all the chickens with the expenses of RM0.50 per chicken in a day. Calculate the total expenses of the food spent by Mr Suhaimi if he starts with 1 500 chickens until he is left with 300 chickens.

Solution

1. Understanding the problem

- Find the total expenses of chicken food until there are 300 chickens left.

2. Planning a strategy

- Form an arithmetic progression series with a first term, a and common difference d, until the last term, 300.
- Determine the number of days for Encik Suhaimi to be left with 300 chickens by using the formula $T_n = a + (n - 1)d$.
- Determine the total expenses of food when there are 300 chickens left by using the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$.

3. Implementing the strategy

Arithmetic progression:
1 500, 1 300, 1 100, …, 300
First term = 1 500
Common difference = –200

$$T_n = a + (n - 1)d$$
$$300 = 1\,500 + (n - 1)(-200)$$
$$300 = 1\,700 - 200n$$
$$200n = 1\,400$$
$$n = 7$$

On the 7th day, the number of chickens left is 300.

$$S_7 = \frac{7}{2}[2(1\,500) + (7 - 1)(-200)]$$
$$= 6\,300$$

The total expenses of food is
= 6 300 × RM0.50
= RM3 150

4. Making a conclusion

$n = 7$, $T_7 = 1\,500 + (7 - 1)(-200)$
$= 300$

CHAPTER 5

5.1.4

Self Practice 5.4

1. Mr. Tong ordered 1 000 Form 4 Mathematics textbook to be sold at his shop. He estimated that 10 books would be sold on the first day, 14 books on the second day and 18 books on the third day and the following day with the same increment.
   (a) Calculate the number of days for Mr. Tong to sell all the books.
   (b) Calculate the increment of the books to sell each day in order for all the books to be sold in 10 days.

2. A wire with length of 240 cm is cut into 15 pieces such that the length of each piece follows the arithmetic progression. The longest piece of the wire is 30 cm.
   (a) Calculate the length of the shortest wire.
   (b) Find the difference between two consecutive wires.

Intensive Practice 5.1

Scan the QR code or visit bit.ly/2p3S7zY for the quiz

1. Determine whether the following sequence is an arithmetic progression and give justification for your answer.
   (a) $-32, -17, -2, 13$
   (b) $8.2, 5.7, 3.2, 1.7, -0.8$

2. For each of the following arithmetic progression, find the n^{th} term as stated in the bracket.
   (a) $-12, -9, -6, \ldots$ [9^{th} term]
   (b) $\frac{1}{3}, -\frac{1}{3}, -1, \ldots$ [15^{th} term]

3. Determine the number of terms for each of the following arithmetic progression.
   (a) $-0.12, 0.07, 0.26, \ldots, 1.97$
   (b) $x, 3x + y, 5x + 2y, \ldots, 27x + 13y$

4. Find the sum of arithmetic progression $-23, -17, -11, \ldots$ which
   (a) has 17 terms,
   (b) has $2n$ terms, in terms of n
   (c) the last term is 121.

5. Given that $S_n = 2n^2 - 5n$, find
   (a) the first term,
   (b) the 9^{th} term,
   (c) the sum of the 4^{th} term to the 8^{th} term.

6. The 2^{nd} term of an arithmetic progression is $\frac{1}{2}$ and the sum of first 14^{th} terms is -70. Find
   (a) the common difference,
   (b) the last term.

7. Yui Ming received offers to work in two companies with the following income.

   Company A: Monthly income is RM3 500 and the increment is RM20 per month.
   Company B: Annual income is RM46 000 and the increment is RM1 000 per year.

   Yui Ming plans to work for 3 years. Which company is more suitable for her in order to receive maximum income in 3 years? Show your calculation and calculate the difference of the total income between the two companies.

5.2 Geometric Progressions

Identifying geometric progressions

There is a famous legend regarding the invention of chess related to series. According to the legend, a king from India wanted to meet the chess inventor to give recognition because this invention was interesting. The chess inventor only requested wheat to be given to him according to the following calculation:

> 1 grain of wheat in the first grid, 2 grains of wheat in the second grid, 4 grains of wheat in the third grid and so on until the last grid.

When the chess board was filled, the total amount of grains given to the chess inventor was 1.84×10^{19}, which is equivalent to 1.2 metric tons. The calculation of the amount of grains can be obtained using geometric progression.

INQUIRY 4 In groups 21st Century Learning

Aim: To identify geometric progression

Instruction:

1. Read the situation below carefully.

> There are various bacteria in our surroundings. Bacteria can be found on unhygienic food, human and animal intestines. Bacteria can reproduce very fast and lead to diseases such as diarrhoea. The rate at which a bacterium reproduces is double at the same rate, which is every 20 minutes: a bacterium will become two, two bacteria will become four and so on. If a human intestine has 2 million bacteria, he/she will be infected with diarrhoea.

2. Let's say if a type of food has only one bacterium. If you consume the food, estimate the time taken for you to be infected with diarrhoea.
3. The table below shows the number of bacteria reproduced. One box represents the reproduction of the bacteria in 20 minutes. Complete the table below until the number of bacteria leads to diarrhoea.

1	2	$4 = 2^2$	$8 = 2^3$			

5.2.1

4. How long does it take for you to be infected with diarrhoea?
5. Determine the method to get the number of bacteria every 20 minutes from the previous 20 minutes. Is the value a constant?
6. Use the *GeoGebra* software and draw a graph to represent the number of bacteria increasing with time.
7. Discuss with your group members about the results obtained and record the results obtained on a piece of paper.
8. Each group moves to another group to compare the results obtained.

From the results of Inquiry 4, it is found that the ratio between any two consecutive terms is a fixed number. So, the progression is called **geometric progression**.

Geometric progression is a sequence of numbers where each term is obtained by multiplying a constant with the previous term.

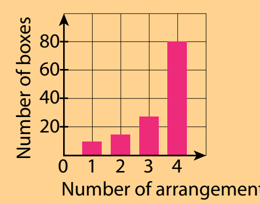
If T_1, T_2, T_3, …, T_n, is the first n terms of a geometric progression. The ratio of the two consecutive terms is called the common ratio, r.

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = \frac{T_n}{T_{n-1}} \leftarrow r \neq \frac{T_1}{T_2} \neq \frac{T_2}{T_3} \neq \ldots \neq \frac{T_{n-1}}{T_n}$$

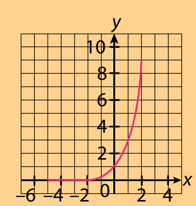
MATHEMATICS POCKET

The graph for geometric progression is quite similar to exponential function graph. Geometric progression graph is discrete while exponential function graph is parallel.

Graph of geometric progression



Graph of exponential function



CHAPTER 5

Example 10

Determine whether the progression below is a geometric progression. Give your justification.
(a) 5, 15, 45, 135, …
(b) 0.1, 0.2, 0.3, …

Solution

(a) $r_1 = \frac{15}{5} = 3, r_2 = \frac{45}{15} = 3, r_3 = \frac{135}{45} = 3$

This progression is a geometric progression because the common ratio, r is the same.

(b) $r = \frac{0.2}{0.1} = 2, r = \frac{0.3}{0.2} = \frac{3}{2}$

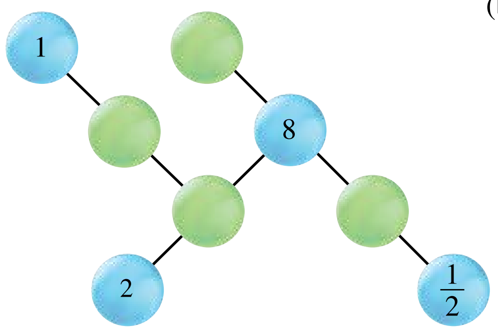
This progression is not a geometric progression because the common ratio, r is different.

Self Practice 5.5

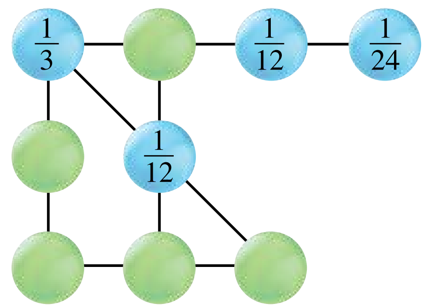
1. Determine whether the following sequence is a geometric progression. Justify your answer.
   (a) $120, 40, \frac{40}{3}, \ldots$
   (b) $0.03, 0.003, 0.0003, \ldots$
   (c) $x + 1, 2x, 5x + 12, 12x, \ldots$

2. Complete the network diagram below, given that the relationship of each of the following network is a consecutive term in a geometric progression.

(a)



(b)



3. Given that $x - 2, x + 1, 4x + 4$ are three consecutive terms in a geometric progression, state the positive value of x. Hence, list the first three terms and state the common ratio.

Deriving the formula of the n^{th} term, T_n, of geometric progressions

INQUIRY 5 In groups

Aim: To derive the formula of the n^{th} term, T_n, of geometric progressions

Instruction:

1. Consider the geometric progression 2, 6, 18, 54, ... with the first term, a and the common ratio, r.
2. Discuss with your group members and complete the table below.

Term	Value of term	Method to obtain the value of term	Formulae
T_1	2	$2(3)^{1-1} = 2(3)^0$	a
T_2	6	$2(3)^{2-1} = 2(3)^1$	$ar = ar^{2-1}$
T_3	18		
T_4	54		
T_5			
⋮	⋮	⋮	⋮
T_n			

3. Derive a formula of the n^{th} term of geometric progressions.

5.2.1 5.2.2

From the results of Inquiry 5, we can notice that the value of each term in this geometric progression can be obtained by using the formula below.

$$T_n = ar^{n-1}$$

Such that a is the first term, r is the common ratio and n is the number of terms.

Example 11

(a) Find the common ratio and the 5th term of the geometric progression 4, –20, 100, –500, …

(b) Find the common ratio and the 7th term of the geometric progression $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

Solution

(a) First term, $a = 4$

Common ratio, $r = \frac{-20}{4} = -5$

$T_5 = 4(-5)^{5-1}$
$= 2\,500$

(b) First term, $a = 2$

Common ratio, $r = \frac{2}{3} \div 2 = \frac{1}{3}$

$T_7 = 2\left(\frac{1}{3}\right)^{7-1}$
$= \frac{2}{729}$

Example 12

Find the number of terms of the geometric progression $-\frac{25}{3}, \frac{5}{3}, -\frac{1}{3}, \ldots, \frac{1}{9\,375}$.

Solution

First term, $a = -\frac{25}{3}$, common ratio $r = \frac{5}{3} \div \left(-\frac{25}{3}\right) = -\frac{1}{5}$

$$T_n = ar^{n-1}$$
$$\frac{1}{9\,375} = \left(-\frac{25}{3}\right)\left(-\frac{1}{5}\right)^{n-1}$$
$$-\frac{1}{78\,125} = \left(-\frac{1}{5}\right)^{n-1}$$
$$\left(-\frac{1}{5}\right)^{7} = \left(-\frac{1}{5}\right)^{n-1}$$
$$7 = n - 1$$
$$n = 8$$

Thus, the number of terms is $n = 8$.

Example 13

An open stadium has 20 chairs in the first row. The number of chairs in the next row is one and a half times the number of chairs of the previous row.

(a) Calculate the maximum number of chairs in the 10th row.

(b) Which row has at least 505 chairs?

Solution

(a) First term, $a = 20$
Common ratio, $r = 1.5$
Sequence in geometric progression:
20, 30, 45, …
$T_{10} = 20(1.5)^9$
$= 768.9$

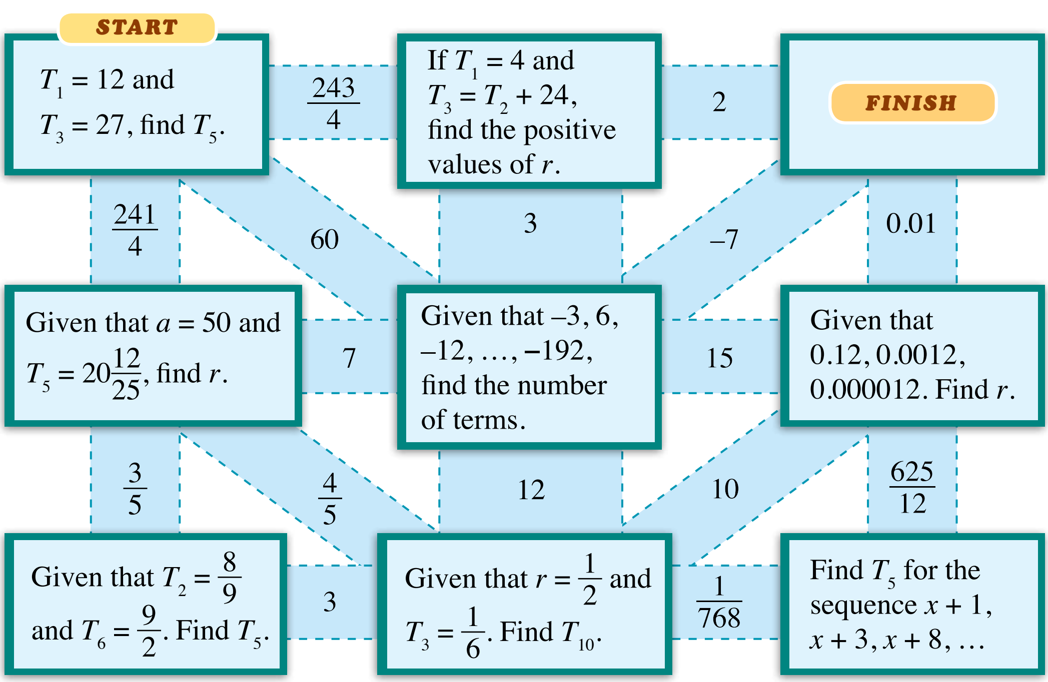
Thus, the maximum number of chairs in the 10^{th} row is 768.

(b)
$$20(1.5)^{n-1} \geqslant 505$$
$$(1.5)^{n-1} \geqslant \frac{505}{20}$$
$$(n-1)\log 1.5 \geqslant \log \frac{505}{20}$$
$$n - 1 \geqslant \frac{\log \frac{505}{20}}{\log 1.5}$$
$$n \geqslant 7.96 + 1$$
$$n \geqslant 8.96$$

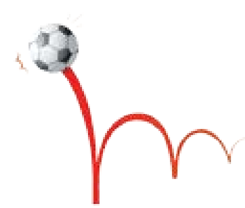
Thus, the 9^{th} row has at least 505 chairs.

Self Practice 5.6

1. Find the way to the "FINISH" box by choosing the correct answer.



2. The diagram on the right shows a ball bouncing on the floor. The maximum height of the bounced of the ball is 3 m and the height of each bounce is 95% of the previous bounce. From which bounce onwards will the height be less than 1 m?



CHAPTER 5

Deriving the formula of sum of the first *n* terms, S_n, of geometric progressions

Consider a geometric progression with the following terms:

$$a, ar, ar^2, ar^3, \ldots, ar^{n-2}, ar^{n-1}$$

Let's say the sum of the first *n* terms is S_n.

Hence, $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$ … ①

① × r: $rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$ … ②

A ① – ②:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$
$$-\ rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

All the terms in between a and ar^n is eliminated.

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

Commonly used when $|r| < 1$

B If ② – ①:

$$rS_n - S_n = ar^n - a$$
$$S_n(r - 1) = a(r^n - 1)$$
$$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

Commonly used when $|r| > 1$

In a geometric progression, the n^{th} term is also calculated by deducting the sum of $(n - 1)^{th}$ term from the sum of n^{th} term. For example, given that the geometric progression is 1, –3, 9, –27, … the 5^{th} term can be calculated by deducting the sum of first 4 terms from the sum of first 5 terms, which is $T_5 = S_5 - S_4$. Therefore, the formula to find T_n by using the sum of terms can be written as:

$$T_n = S_n - S_{n-1}$$

Smart TIPS

- $|r| < 1$ can be written as $-1 < r < 1$.
- $|r| > 1$ can be written as $r < -1$ and $r > 1$.

MATHEMATICS POCKET

1, 2, 4, … is a geometric progression whereas 1 + 2 + 4 + … is a geometric series.

Example 14

Given that a geometric series is 1 + 5 + 25 + 125 + 625 + …

(a) Find the sum of first 10 terms.

(b) Find the value of *n* where S_n = 3 906.

Solution

(a) First term, $a = 1$

Common ratio, $r = 5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Use this formula because $|r| > 1$

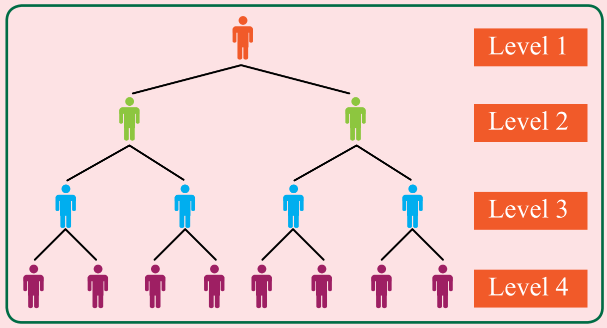
$$S_{10} = \frac{1(5^{10} - 1)}{5 - 1}$$
$$= 2\ 441\ 406$$

(b)

$$S_n = 3\ 906$$
$$\frac{1(5^n - 1)}{5 - 1} = 3\ 906$$
$$5^n - 1 = 15\ 624$$
$$5^n = 15\ 625$$
$$n \log 5 = \log 15\ 625$$
$$n = \frac{\log 15\ 625}{\log 5}$$
$$= 6$$

Example 15

A health product company has planned a marketing strategy. Each member has to promote the company products by getting two downlines.
(a) Show that the number of members in each level follows a geometric progression.
(b) If there are 9 levels in the marketing strategy, find the total number of members involved in promoting the product.



Solution

(a) The number of members in each level can be written as 1, 2, 4, 8, …

$r = \frac{2}{1} = \frac{4}{2} = 2$

Since $r = 2$, thus, the number of members in each level follows a geometric progression.

(b) When $n = 9$, $S_9 = 1 + 2 + 4 + 8 + \dots + T_9$

Use $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_9 = \frac{1(2^9 - 1)}{2 - 1}$

$= 511$

The total number of members involved in promoting the product is 511 members.

BRAINSTORMING

By using $S_n = \frac{a(r^n - 1)}{r - 1}$ and $S_{n-1} = \frac{a(r^{n-1} - 1)}{r - 1}$, prove that $T_n = ar^{n-1}$.

Self Practice 5.7

1. Find the sum of each of the following.
(a) 0.02, 0.04, 0.08, …, T_{12}
(b) $p, p^3, p^5, \dots, p^{21}$, in terms of p
(c) $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \dots$ till the first 15 terms

2. Find the number of terms if the sum of geometric progression 3 500, 700, 140, … is 4 368.

3. A squared paper is cut into 4 equal-sized squares. Each portion is then cut again into 4 smaller equal-sized portion. This process is repeated on each of the small portion square.
(a) Show that the number of squares cut forms a geometric progression.
(b) Find the total squares obtained if the process is repeated for 6 times.

5.2.3

Determining the sum to infinity of geometric progressions

INQUIRY 6 In groups

Aim: To determine the sum to infinity of geometric progressions

Instruction:

1. Consider the geometric progression 64, 32, 16, ...
2. Complete the table on the right for the value of r^n and S_n.
3. Discuss with your group members about your observation on these two values when n increases.
4. Derive a conclusion for $S_n = \frac{a(1 - r^n)}{1 - r}$ when n increases to infinity. Then express S_∞ in terms of a and r.
5. One of the group members will present their findings in front of the class and the members from the other groups will ask questions.
6. Other groups will take turns to do their presentation.

n	r^n	S_n
1		
2		
3		
4		
5		
10		
20		
100		
200		

From the results of Inquiry 6, when the value of n increases and get closer to infinity ($n \to \infty$), the value or r^n will decrease and get closer to zero ($r^n \to 0$) whereas the value of S_n will get closer to $\frac{a}{1 - r}$ $\left(S_n \to \frac{a}{1 - r}\right)$. Hence, the sum to infinity of geometric progressions is

$$S_\infty = \frac{a}{1 - r}, \text{ where } |r| < 1$$

Example 16

Find the sum to infinity of geometric progressions 45, 9, 1.8, …

Solution

$a = 45,\ r = \frac{9}{45} = \frac{1}{5}$

$$S_\infty = \frac{45}{1 - \frac{1}{5}}$$
$$= 56\frac{1}{4}$$

QR

Proof of Pythagoras theorem using sum to infinity of geometric progressions.

bit.ly/2nA0ra4

Example 17

The sum to infinity of geometric progressions is $31\frac{1}{2}$ and the sum of first two terms is 28. Find the common ratio.

Solution

$$S_\infty = 31\frac{1}{2}$$

$$\frac{a}{1-r} = \frac{63}{2}$$

$$a = \frac{63}{2}(1-r) \quad \dots ①$$

$$a + ar = 28$$

$$a(1+r) = 28 \quad \dots ②$$

$$② \div ①, \quad \frac{a(1+r)}{a} = \frac{28}{\frac{63}{2}(1-r)}$$

$$(1+r)(1-r) = \frac{8}{9}$$

$$1 - r^2 = \frac{8}{9}$$

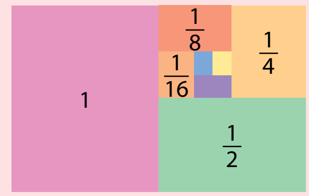
$$r^2 = \frac{1}{9}$$

$$r = \frac{1}{3} \quad \text{or} \quad r = -\frac{1}{3}$$

Mind Challenge

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$

Look at the diagram below and make a conclusion.



Use the similar diagram and prove that

$2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots = 4.$

Example 18

Express the repeating decimals 0.56363… in the form of sum to infinity of geometric progressions. Hence, express the number in the simplest form of fraction.

Solution

$$\begin{aligned} 0.56363\dots &= 0.5 + 0.063 + 0.00063 + 0.0000063 + \dots \\ &= 0.5 + (0.063 + 0.00063 + 0.0000063 + \dots) \\ &= 0.5 + S_\infty \\ &= \frac{1}{2} + \frac{0.063}{1 - 0.01} \\ &= \frac{1}{2} + \frac{7}{110} \\ &= \frac{31}{55} \end{aligned}$$

MATHEMATICS POCKET

The repeating decimals such as 0.56363… can be written as $0.5\dot{6}\dot{3}$.

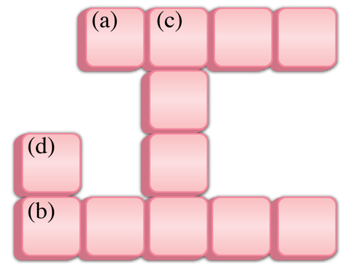
Smart TIPS

0.063 + 0.00063 + 0.0000063 + … is a geometric series with $a = 0.063$ and $r = 0.01$.

5.2.4

Self Practice 5.8

1. Complete the following crossword puzzle.



Horizontal:

(a) Find the sum to infinity of geometric progressions
$1\,500, 500, 166\frac{2}{3}, \ldots$

(b) Wilson loaned RM15 000 to buy a motorcycle. Every year, he managed to reduce 50% of his total loan. Find the maximum amount paid by Wilson.

Vertical:

(c) Given that the sum to infinity is 4 480 and the common ratio is $\frac{1}{2}$, find the first term of this geometric progressions.

(d) 4.818181… can be written in the form of $\frac{h}{11}$, find the value of h.

Solving problems involving geometric progressions

Example 19

A telecommunication company managed to sell 0.5 million smartphones in the year 2015. Every year, the sales of the smartphone increases by 4%.

(a) Find the total number of smartphones sold from the year 2015 to the year 2020.

(b) If 33% of the smartphones sold from the year 2017 to the year 2020 is 5-inched phones and 14% are 6-inched phones, calculate the total number of 5-inched and 6-inched phones.

Solution

(a) Geometric progression (in million): $0.5, 0.5(1.04), 0.5(1.04)^2, \ldots$

$a = 0.5$ million, $r = 1.04$

$$S_6 = \frac{0.5(1.04^6 - 1)}{1.04 - 1}$$
$$= 3.316 \text{ million}$$

(b) The total number of smartphones from year 2017 to year 2020.

$$S_6 - S_2 = \frac{0.5(1.04^6 - 1)}{1.04 - 1} - \frac{0.5(1.04^2 - 1)}{1.04 - 1}$$
$$= 3.316 \text{ million} - 1.02 \text{ million}$$
$$= 2.296 \text{ million}$$

The number of 5-inched smartphones:

$$\frac{33}{100} \times 2.296 \text{ million} = 0.758 \text{ million}$$

The number of 6-inched smartphones:

$$\frac{14}{100} \times 2.296 \text{ million} = 0.321 \text{ million}$$

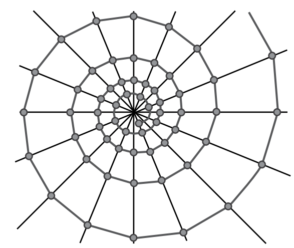
Total number of smartphones = 0.758 million + 0.321 million
= 1.079 million

Thus, the total number of 5-inched and 6-inched smartphones sold are 1.079 million.

Self Practice 5.9

1. A wire is cut into a few pieces, in which $10x$ cm, $(4x + 20)$ cm and $(3x - 10)$ cm are the three consecutive pieces of a geometric progression.
   (a) Find the longest piece if $10x$ is the 2nd longest term.
   (b) If the wire is cut into sum to infinity pieces, find the maximum length of the wire, in m.

2. The diagram on the right shows the pattern of a spider web. The perimeter of each semicircle follows geometric progression with the smallest radius r cm and each subsequent radius increases 40%.
   (a) Form the first three terms for the perimeter of the semicircle in terms of r.
   (b) Find the total length of semicircle, in m, if the spiderweb has 15 semicircles and each radius is 2 cm.



Intensive Practice 5.2

Scan QR code or visit bit.ly/2VrxSZ5 for the quiz

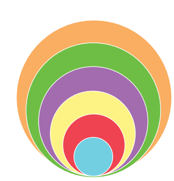
1. Calculate the number of terms and the sum for each of the following geometric progression.
   (a) $-1, 3, -9, \ldots, 2\,187$
   (b) $\log x^{-1}, \log x^{-2}, \log x^{-4}, \ldots, \log x^{-64}$
   (c) $0.54, 0.0054, 0.000054, \ldots\ 5.4 \times 10^{-17}$
   (d) $3, \frac{3}{2}, \frac{3}{4}, \ldots\ \frac{3}{64}$

2. Given that the geometric progression $4.5, -9, 18, \ldots$ Find the number of terms for this geometric progression in order for the sum to be 769.5.

3. Three consecutive terms of a geometric progression is x, $2x + 3$ and $10x - 3$. Find
   (a) all the possible values of x,
   (b) the 6th term if $x < 0$.

4. The diagram shows a few triangles. Given that the area of the triangles follows geometric progression such that the area of the third triangle is 36 cm^2 and the sum of area of the third and fourth triangle is 54 cm^2. Find
   (a) the common ratio and the area of the first triangle,
   (b) the sum of the third triangle to the tenth triangle.

5. The diagram shows a few circles with the same centre. The perimeter of each circle follows the geometric progression. Given that the n^{th} perimeter is $T_n = 3^{8-n}$ cm, find
   (a) the common ratio,
   (b) the sum of three consecutive perimeter after the second largest perimeter.

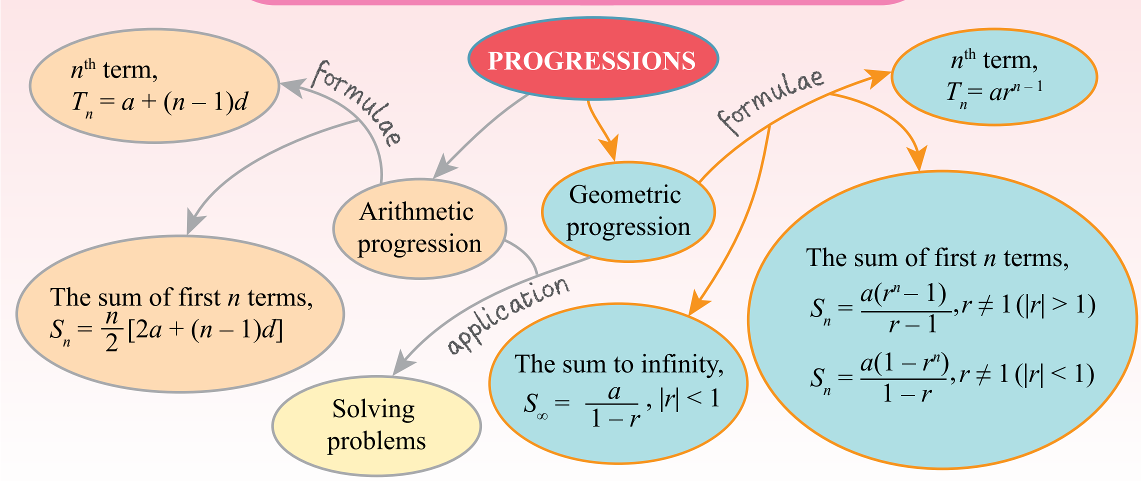


6. There are three children with their mass arranged according to the descending order of a geometric progression. The sum of mass of three of them is seven times the mass of the lightest child. Find the common ratio and the mass of the child whose weight is second heaviest if the heaviest child is 14.5 kg.

CHAPTER 5

5.2.5

SUMMARY OF CHAPTER 5



WRITE YOUR JOURNAL

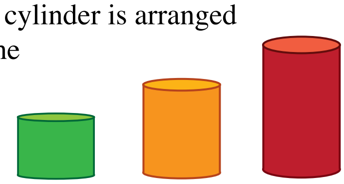
Construct a graphical information on the difference between the arithmetic progression and the geometric progression. Then, think of a situation in your daily life that applies these two progressions and solve it.

MASTERY PRACTICE

1. $-2x - 1$, $3x + 2$ and $9x + 3$ are three consecutive terms of arithmetic progression. Find PL1
(a) common difference,
(b) the first term if $3x + 2$ is the third term.

2. The 9^{th} term of an arithmetic progression is $21 + 3p$ and the sum of the first three terms is $9p$. Find the common difference. PL2

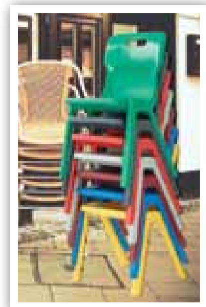
3. The diagram shows three cylinders such that the volume of each cylinder is arranged according to arithmetic progression. The sum of the volume of the first and third cylinder is 24 cm^3 and the volume of the fifth cylinder is 36 cm^3. PL3



(a) Find the volume of the smallest cylinder.
(b) Calculate the sum of volume for the first 9 cylinders.

4. The 3^{rd} term of a geometric progression is 30 and the sum of the 3^{rd} and 4^{th} terms is 45. Find PL2
(a) the first term and the common ratio,
(b) the sum to infinity.

5. The diagram shows the arrangement of a few chairs. The height of each chair is 80 cm. When the chairs are arranged, there is a 4 cm gap in between two chairs. The arranged chairs will be kept in the store. PL4
   (a) Find the maximum number of chairs that can be arranged if the height of the store is 3 m.
   (b) 13 chairs will be kept in the store with the condition that the first stack will have the maximum number of chairs and the arrangement of chairs for subsequent stacks decreases 2. Calculate the total number of chair kept in the store.



6. Encik Muslim starts to save RM14 000 into his new born baby's account. The bank offers a 5% interest yearly. Encik Muslim hopes that his child's saving will reach RM30 000 when his child becomes 18 years old. PL4
   (a) Do you think he can obtain RM30 000 when his child becomes 18 years old? Show your calculations.
   (b) If the interest decreases to 3% per year after 10 years, calculate the total savings when Encik Muslim's child is 18 years old. Can the savings of his child reach RM30 000?

7. Shahrul has a toy car collection which he collects every month. The number of toy cars increases every month according to geometric progression. The total number of toy cars in the first four months is ten times the total number of toy car in the first month. PL5
   (a) If r represents the common ratio, show that $r^4 - 10r^2 + 9 = 0$. Hence, find the positive values of r.
   (b) Calculate the expenses paid by Shahrul in the 6 months if he started to buy 2 toy cars and the average price of a toy car is RM7.50.

Exploring MATHEMATICS

1. Prepare two piggy banks.
2. In 10 days, put money into the piggy bank by following these rules:

> First piggy bank:
> Start putting 50 cents into the piggy bank on the first day, RM1 on the second day RM1.50 on the third day and so on. Each day, the total savings increases 50 cents.

> Second piggy bank:
> Start putting 10 cents into the piggy bank on the first day, 20 cents on the second day 40 cents on the third day and so on. Each day, the total savings increases twice the amount compared to the previous day.

3. Record the total savings after 10 days.
4. Observe the relationship between the total savings with progression.
5. Prepare a report on the relation between the arithmetic progression and geometric progression with the total amount of your savings.

CHAPTER 6

Linear Law

What will be learnt?

- Linear and Non-Linear Relations
- Linear Law and Non-Linear Relations
- Application of Linear Law

List of Learning Standards

bit.ly/2VtVKLE

KEYWORDS

- Linear relation — *Hubungan linear*
- Non-linear relation — *Hubungan tak linear*
- Inspection method — *Kaedah pemerinyuan*
- Line of best fit — *Garis lurus penyuaian terbaik*

String art is a type of art which uses strings or ropes to form geometric patterns. String art applies the use of straight lines to form patterns which are not straight lines.

Did you Know?

Leonardo Bonacci or known as Fibonacci was a famous mathematician in Italy in the 13th century. He had discovered a concept such that the ratio of distance between the tip of nose and tip of chin and from the cheek to the tip of chin gave a value which is known as the Golden Ratio. The ratio of readings can be measured and represented through a straight line graph involving two variables.

For further information:

bit.ly/2M4jLFO

SIGNIFICANCE OF THIS CHAPTER

In order to relate two variables, a straight line will help us to get the value of a constant. When a straight line is drawn from the results of an experiment, sometimes the data obtained do not produce a perfect straight line. Therefore, the data will be represented by a line of best fit.

Scan the QR code to watch the video showing the methods of making string art.

bit.ly/2TWUXAY

6.1 Linear and Non-Linear Relations

Differentiating linear and non-linear relations

INQUIRY 1 In pairs

Aim: To differentiate between linear and non-linear relations based on tables of data and graphs

Instructions:

1. Complete the table based on the given equation.
   (a) $y = 2x^2 - 5x + 8$

x	–3	–2	–1	0	1	2	3	4
y								

   (b) $y = x + 4$

x	–4	–3	–2	–1	0	1	2	3
y								

2. Draw the graph of y against x based on the values obtained in both tables for each of the equations.
3. Based on the graphs drawn, compare the shapes of the graphs for both equations. What do you observe?

From the results of Inquiry 1, we can conclude that:

The graph which forms a straight line is a linear relation whereas the graph which does not form a straight line is a non-linear relation.

A linear graph can be obtained from a non-linear graph when the variables of the X-axis or Y-axis or both are changed.

FLASHBACK

For a linear graph, $Y = mX + c$, X represents the variable on the horizontal axis, Y represents the variable on the vertical axis, m represents the gradient and c represents the Y-intercept.

Example 1

Draw the graph of Y against X based on each of the following tables of data and hence determine which graph is a graph of linear relation? Give your reason.

(a)

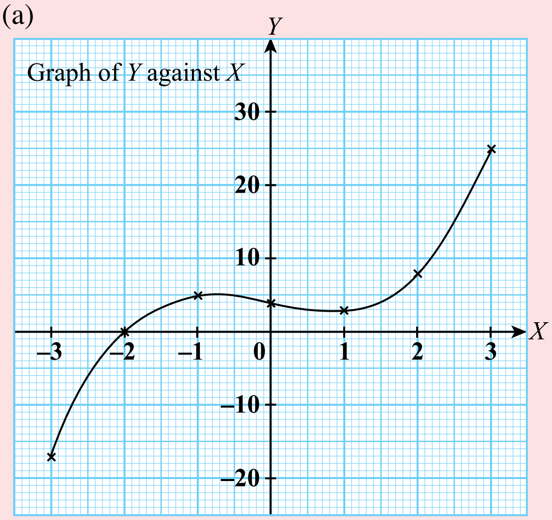
X	–3	–2	–1	0	1	2	3
Y	–17	0	5	4	3	8	25

(b)

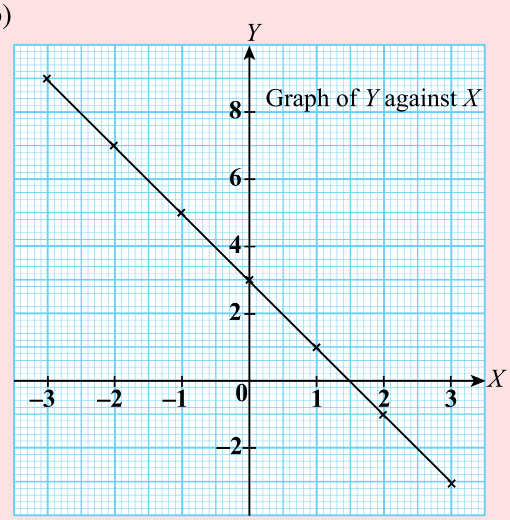
X	–3	–2	–1	0	1	2	3
Y	9	7	5	3	1	–1	–3

Solution

(a)



(b)



Graph (a) is a graph of non-linear relation because the graph obtained is a curve.
Graph (b) is a graph of linear relation because the graph obtained is a straight line.

Self Practice 6.1

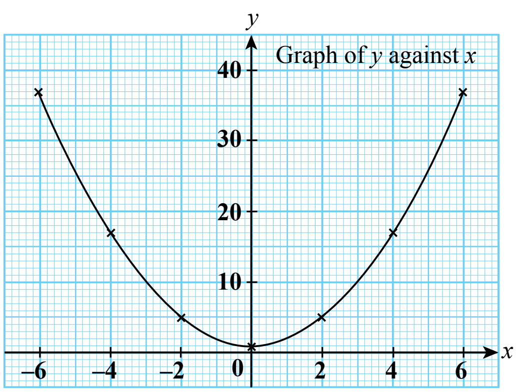
1. The diagrams below show two graphs plotted by using the given values in the respective tables for the equation $y = x^2 + 1$. Which graph shows a graph of linear relation? State your reasons.

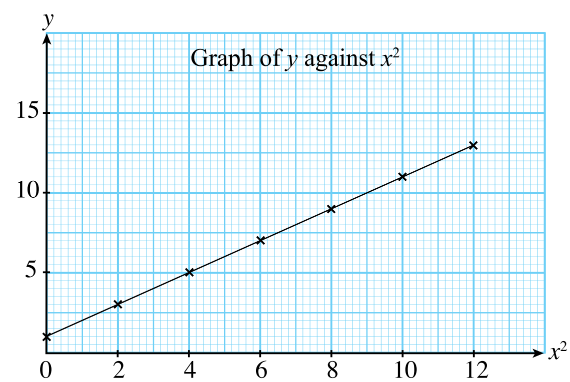
(a)

x	–6	–4	–2	0	2	4	6
y	37	17	5	1	5	17	37

(b)

x^2	0	2	4	6	8	10	12
y	1	3	5	7	9	11	13





2. Draw the graph of Y against X based on the given values in the following tables.

(a)

X	1	3	5	7	9	11
Y	3.16	5.50	9.12	16.22	28.84	46.77

(b)

X	2	4	6	10	12	14
Y	0.5	0.7	0.9	1.3	1.5	1.7

Which graph shows a graph of linear relation? State your reasons.

6.1.1

Drawing lines of best fit for graphs of linear relations

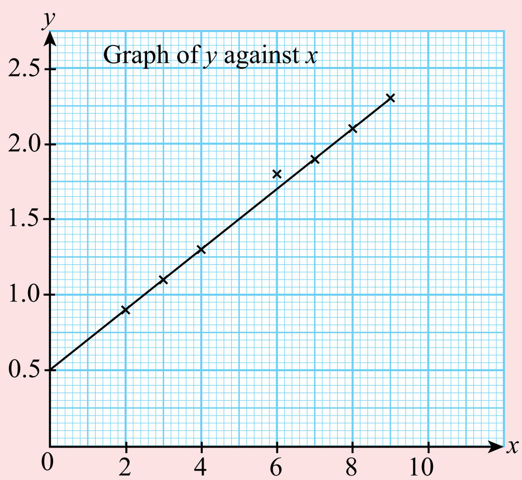
A line of best fit is a straight line that joins most of the points plotted on the graph. The points which are not on the line of best fit must be distributed evenly on both sides of the straight line.

Example 2

The table on the right shows the values obtained from an experiment involving two variables, x and y. Plot the graph of y against x, by using suitable scales on the x-axis and y-axis. Hence, draw the line of best fit.

x	2	3	4	6	7	8	9
y	0.9	1.1	1.3	1.8	1.9	2.1	2.3

Solution



INQUIRY 2

In pairs

21st Century Learning

Aim: To draw the line of best fit using digital technology

Instructions:

1. Draw the straight line graph based on the following data values.

x	1	2	3	4	6	7
y	3	5	6	8	10	11

2. Then, enter the values in the table provided into the Desmos software by using the same data values as given in the table above.
3. Follow the diagrammatic steps to draw the line of best fit by scanning the QR code on the right.
4. Compare the line of best fit obtained in the Desmos software with the graph drawn.

QR

Steps to draw line of best fit using *Desmos* application.

bit.ly/33jaKi3

CHAPTER 6

From the results of Inquiry 2, it is observed that:

The straight line obtained from the graph drawn is the same as the straight line drawn using the *Desmos* software. The line is the line of best fit.

Self Practice 6.2

1. The following table shows the values obtained from an experiment involving two variables, x and y.

x	5	10	15	20	25	30
y	8	14.5	18	23	26.5	33

Plot the graph of y against x, by using suitable scales on the x-axis and y-axis. Hence, draw the line of best fit.

2. An experiment was carried out to determine the relation between extension of spring, L and mass of load, m, which was hung at the end of the spring. The following table shows the results from the experiment.

m (g)	20	40	60	80	100	120
L (cm)	0.65	1.25	1.80	2.40	2.95	3.55

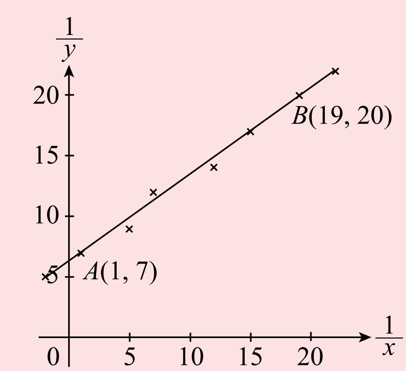
Plot the graph of L against m, by using suitable scales on the m-axis and L-axis. Hence, draw the line of best fit.

Forming equations of lines of best fit

The equation of straight lines can be written in the form of $Y = mX + c$, if the gradient, m and Y-intercept, c are known or can be determined by using any two points on the straight line.

Example 3

The graph below shows part of a straight line obtained by plotting $\frac{1}{y}$ against $\frac{1}{x}$. Express y in terms of x.



Solution

$$\text{Gradient, } m = \frac{20 - 7}{19 - 1} = \frac{13}{18}$$

$$Y = mX + c$$

$$\frac{1}{y} = m\left(\frac{1}{x}\right) + c$$

$$20 = \left(\frac{13}{18}\right)(19) + c$$

$$c = \frac{113}{18}$$ ← *Y*-intercept

Thus, $\frac{1}{y} = \frac{13}{18}\left(\frac{1}{x}\right) + \frac{113}{18}$ ← Equation of straight line

$$y = \frac{18x}{13 + 113x}$$

Example 4

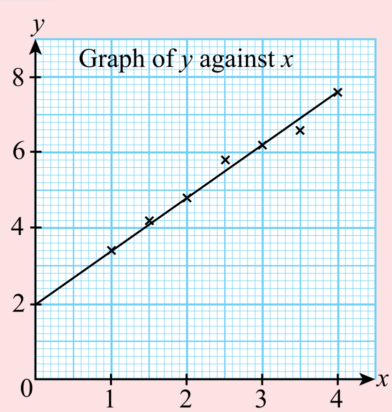
The following table shows the experimental values of two variables, x and y.

x	1	1.5	2	2.5	3	3.5	4
y	3.4	4.2	4.8	5.8	6.2	6.6	7.6

(a) Plot the graph of y against x, by using a scale of 1 cm to 1 unit on the x-axis and 1 cm to 2 units on the y-axis. Hence, draw the line of best fit.
(b) From the graph, find the y-intercept and gradient of the line of best fit.
(c) Determine the equation of the line of best fit.

Solution

(a)



(b) From the graph, y-intercept, $c = 2$

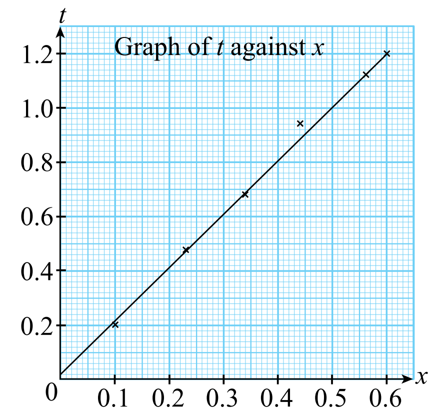
$$\text{gradient, } m = \frac{7.6 - 2}{4 - 0} = 1.4$$

(c) The equation of line of best fit is
$y = 1.4x + 2$.

6.1.3

Self Practice 6.3

1. The graph of line of best fit in the diagram on the right shows the values obtained from an experiment which involves two variables, x and t. Express t in terms of x.



2. The following table shows the experimental values of two variables, x and y.

x	10	20	30	40	50	60
y	16.5	20.0	23.5	27.5	31.5	35.0

(a) Plot the graph of y against x, by using a scale of 2 cm to 10 units on the x-axis and 2 cm to 5 units on the y-axis. Hence, draw the line of best fit.
(b) From the graph, find the y-intercept and gradient of the line of best fit.
(c) Determine the equation of the line of best fit.

Interpreting information based on the lines of best fit

Based on the line of best fit, you can predict the values of variables x or y which are not in the experiment without repeating the experiment. If the values of variable x or y are outside the range of points, you can find the value of the variable by extrapolating the drawn straight line or it can be determined by forming the equation of the straight line.

Example 5

The following table shows the data of two variables, x and $\log_{10} y$, obtained from an experiment.

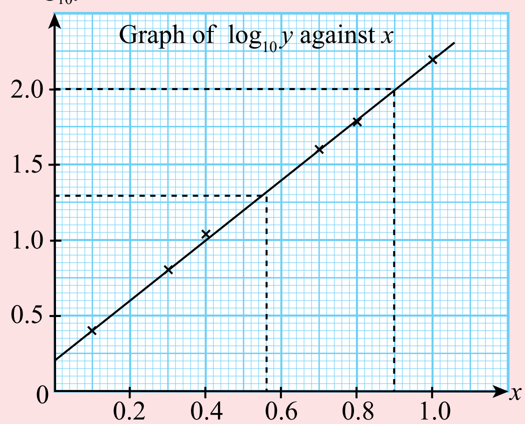
x	0.1	0.3	0.4	0.7	0.8	1.0
$\log_{10} y$	0.40	0.80	1.04	1.60	1.78	2.20

(a) Plot $\log_{10} y$ against x, by using a scale of 1 cm to 0.2 unit on the x-axis and 1 cm to 0.5 unit on the $\log_{10} y$-axis. Hence, draw the line of best fit.
(b) From the graph, find the value of
    (i) $\log_{10} y$ when $x = 0.9$,
    (ii) y when $x = 0$,
    (iii) x when $\log_{10} y = 2$,
    (iv) x when $y = 20$.
(c) Find the equation of the line of best fit.

6.1.3 6.1.4

Solution

(a) $\log_{10} y$



(b) (i) From the graph, when $x = 0.9$, $\log_{10} y = 2$.

(ii) From the graph, when $x = 0$,

$\log_{10} y = 0.2$

$y = 10^{0.2}$

$y = 1.585$

(iii) From the graph, when $\log_{10} y = 2$, $x = 0.9$.

(iv) From the graph, when $y = 20$, $\log_{10} 20 = 1.3$.

Then, $x = 0.56$.

(c) Two points are selected from the graph, they are (0.7, 1.60) and (0.3, 0.80).

$$\text{Gradient, } m = \frac{1.60 - 0.80}{0.7 - 0.3}$$
$$= 2$$

Y-intercept, is 0.2.

Thus, the equation of the line of best fit is $\log_{10} y = 2x + 0.2$.

MATHEMATICS POCKET

- **Gradient** is the rate of change of a variable with respect to another variable.
- ***Y*-intercept** is the *y*-coordinate of the intersection point of a straight line with the *y*-axis.

Self Practice 6.4

1. The following table shows the values of x and y obtained from an experiment.

x	1	2	4	6	8	10	15
y	5.5	7.0	10.5	13.0	15.5	19.0	26.5

(a) Plot y against x, by using a scale of 2 cm to 2 units on the x-axis and 2 cm to 5 units on the y-axis. Hence, draw a line of best fit.

(b) From the graph, find
   (i) the y-intercept,
   (ii) the value of y when $x = 12$,
   (iii) the gradient,
   (iv) the value of x when $y = 15$,

(c) Find the equation of the line of best fit. Hence, calculate the value of y when $x = 28$.

Intensive Practice 6.1

Scan the QR code or visit at bit.ly/2pbnnNb for the quiz

1. The following tables show the experimental data involving variables x and y.

(a)

x	−4	−2	−1	0	1	2
y	3	−3	−3	−1	3	9

(b)

$\frac{1}{x}$	0.80	0.70	0.50	0.40	0.25	0.20
y^2	4.00	4.41	5.20	5.62	6.20	6.40

Draw the graphs based on the data in the tables. Then, determine the graph which shows linear relation and non-linear relation. Give a reason for your answer.

2. Based on an experiment, the values of X and of Y are related as in the following table.

X	20	30	40	50	60	70
Y	108.0	110.4	112.4	114.4	116.8	119.0

Plot the graph of Y against X and draw the line of best fit. Then, write the equation of the line of best fit.

3. The following table shows the readings of two variables, $\log_{10}(x + 1)$ and $\log_{10} y$.

$\log_{10}(x + 1)$	0.18	0.30	0.50	0.60	0.70	0.78
$\log_{10} y$	0.33	0.45	0.64	0.75	0.85	0.93

(a) Plot the graph of $\log_{10} y$ against $\log_{10}(x + 1)$, by using a scale of 2 cm to 0.1 unit on the $\log_{10}(x + 1)$-axis and $\log_{10} y$-axis. Hence, draw the line of best fit.

(b) From the graph, find
  (i) the gradient,
  (ii) the $\log_{10} y$-intercept,
  (iii) the value of x when $\log_{10} y = 0.55$,

(c) Calculate
  (i) the value of y when $x = 2.5$,
  (ii) the value of x when $y = 1.5$.

4. The results of experiment of two variables, x^2 and xy, are shown in the following table.

x^2	5	9	16	25	36	42
xy	12	15.5	22	30	40	45

(a) Plot the graph of xy against x^2, by using a scale of 2 cm to 5 units on X-axis and Y-axis. Hence, draw the line of best fit.

(b) From the graph, find
  (i) the gradient,
  (ii) the Y-intercept,
  (iii) the value of x^2 when $xy = 16.5$,
  (iv) the value of y when $x = 2.5$.

(c) Calculate the value of x when $xy = 100$.

6.2 Linear Law and Non-Linear Relations

Applying linear law to non-linear relations

By using linear law, most of the non-linear relations can be converted to linear relations so that a straight line can be drawn. It is easier to obtain information from the straight line graphs than from the curves.

The non-linear equation $y = ax + \frac{b}{x}$, such that a and b are constants can be converted to linear equation form $Y = mX + c$ by using two methods.

Method 1

$$y = ax + \frac{b}{x}$$

$$y(x) = ax(x) + \frac{b}{x}(x)$$ ← Multiply both sides of the equation by x

$$yx = ax^2 + b$$

$$xy = ax^2 + b$$ ← Compare with $Y = mX + c$

Through comparison, $Y = xy$, $X = x^2$, $m = a$ and $c = b$.

Y	m	X	c
xy	a	x^2	b

Method 2

$$y = ax + \frac{b}{x}$$

$$\frac{y}{x} = \frac{b}{x^2} + \frac{ax}{x}$$ ← Divide both sides of the equation by x

$$\frac{y}{x} = \frac{1}{x^2}(b) + a$$ ← Compare with $Y = mX + c$

Through comparison, $Y = \frac{y}{x}$, $X = \frac{1}{x^2}$, $m = b$ and $c = a$.

Y	m	X	c
$\frac{y}{x}$	b	$\frac{1}{x^2}$	a

Smart TIPS

You have to choose suitable variables for X and Y to change the non-linear equation to the linear form, $Y = mX + c$ such that m is the gradient of the straight line and c is the y-intercept. The variables X and Y must contain variables only and they cannot contain the unknown constants. m and c must contain only constants.

Example 6

Convert the equation $y = pq^x$ such that p and q are constants to the linear form $Y = mX + c$. Hence, identify Y, X, m and c.

Solution

$$y = pq^x$$

$$\log_{10} y = \log_{10} p + x \log_{10} q$$ ← Write both sides of the equation in logarithmic form

$$\log_{10} y = \log_{10} q\,(x) + \log_{10} p$$ ← Compare with $Y = mX + c$

Through comparison, $Y = \log_{10} y$, $X = x$, $m = \log_{10} q$ dan $c = \log_{10} p$

Y	m	X	c
$\log_{10} y$	$\log_{10} q$	x	$\log_{10} p$

6.2.1

Example 7

The table below shows the values of x and y obtained from an experiment. The variables, x and y are related by the equation $3y - px^2 = qx$, such that p and q are constants.

x	1	2	3	5	7	9
y	20	34	48	60	63	36

(a) Convert the equation $3y - px^2 = qx$ to the linear form.

(b) Plot the graph of $\frac{y}{x}$ against x, by using a scale of 1 cm to 2 units on the x-axis and 1 cm to 5 units on the $\frac{y}{x}$-axis. Hence, draw the line of best fit.

(c) From the graph, find the value of p and of q.

Solution

(a) $3y - px^2 = qx$

$\frac{3y}{3x} - \frac{px^2}{3x} = \frac{qx}{3x}$ ← Divide both sides of the equation by $3x$

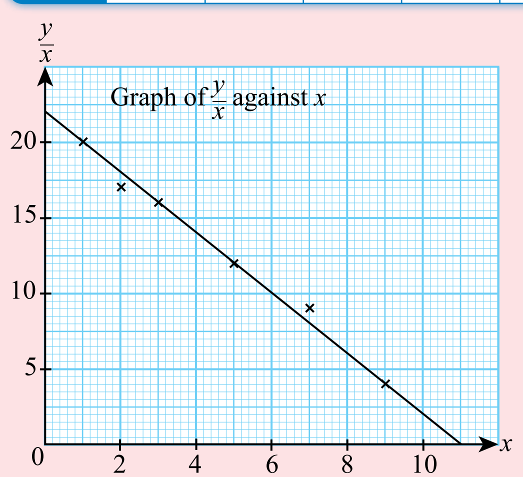
$\frac{y}{x} - \frac{px}{3} = \frac{q}{3}$

$\frac{y}{x} = \frac{p}{3}(x) + \frac{q}{3}$ ← Compare with $Y = mX + c$

Through comparison, $Y = \frac{y}{x}$, $X = x$, $m = \frac{p}{3}$ and $c = \frac{q}{3}$.

(b)

x	1	2	3	5	7	9
$\frac{y}{x}$	20	17	16	12	9	4



(c) From the graph,
y-intercept = 22

$\frac{q}{3} = 22$

$q = 66$

gradient $= \frac{4 - 22}{9 - 0}$

$= -2$

$\frac{p}{3} = -2$

$p = -6$

Mathematics Museum

René Descartes invented the coordinate grids known as Cartesian Diagram. How did the idea of inventing Cartesian Diagram trigger him? He laid down on his bed until late night and observed a housefly on the ceiling of his room. He thought of the best way to illustrate the position of the housefly on the ceiling. He decided to take one corner of the ceiling as the reference point.

For further information:

bit.ly/2oALd54

CHAPTER 6

6.2.1

Example 8

The table below shows the values of x and y obtained from an experiment. The variables x and y are related by the equation $y = \frac{p^x}{q}$, such that p and q are constants.

x	2	4	5	6	7	8	10
y	0.3162	5.0119	100	1 584.89	6 309.57	63 095.73	100 000

(a) Plot the graph of $\log_{10} y$ against x, by using a scale of 1 cm to 2 units on both the $\log_{10} y$-axis and x-axis. Hence, draw the line of best fit.

(b) From the graph, find
  (i) the value of p and of q,
  (ii) the value of y when $x = 3$.

Solution

(a) $y = \frac{p^x}{q}$

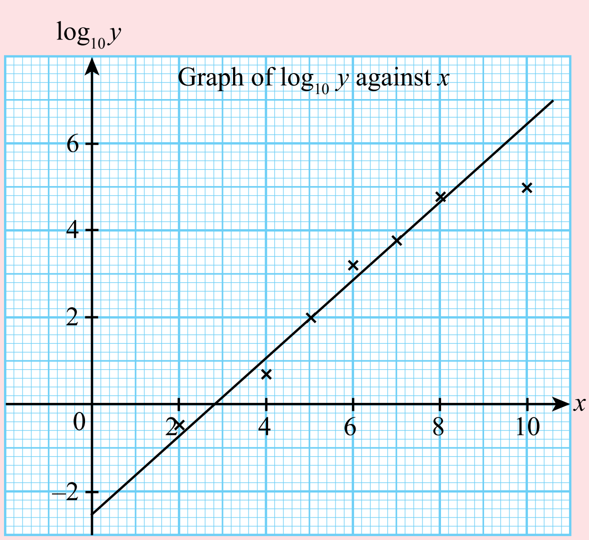
$\log_{10} y = \log_{10} p^x - \log_{10} q$

$\log_{10} y = x \log_{10} p - \log_{10} q$

$\log_{10} y = (\log_{10} p)x - \log_{10} q$ ← Compare with $Y = mX + c$

Through comparison, $Y = \log_{10} y$, $X = x$, $m = \log_{10} p$ and $c = -\log_{10} q$

x	2	4	5	6	7	8	10
$\log_{10} y$	−0.50	0.70	2.00	3.20	3.80	4.80	5.00



(b) (i) $-\log_{10} q = -2.5$

$\log_{10} q = 2.5$

$q = 316.228$

$\log_{10} p = \frac{2.00 - 3.80}{5 - 7}$

$\log_{10} p = 0.9$

$p = 7.943$

(ii) When $x = 3$, $\log_{10} y = 0.2$

$y = 1.585$

CHAPTER 6

6.2.1

Self Practice 6.5

1. Convert the following non-linear equations into the form $Y = mX + c$. Hence, identify Y, X, m and c.

(a) $y = px^2 - q$ (b) $y = hx^2 + x$ (c) $y = \frac{p}{x^2} + q$

2. The table below shows the values of x and y obtained from an experiment. The variables $\sqrt{x}$ and $\frac{1}{y}$ are related by the equation $\frac{1}{y} = p\sqrt{x} + q$, such that p and q are constants.

$\sqrt{x}$	0.70	1.00	1.22	1.45	1.58	1.80
$\frac{1}{y}$	0.62	1.20	1.65	2.00	2.38	2.75

(a) Plot the graph of $\frac{1}{y}$ against $\sqrt{x}$, by using a scale of 1 cm to 0.5 unit on both the $\sqrt{x}$-axis and $\frac{1}{y}$-axis. Hence, draw the line of best fit.

(b) From the graph, find the value of

(i) q, (ii) p, (iii) y when $x = 1.21$.

Intensive Practice 6.2

Scan the QR code or visit bit.ly/2M54EMF for the quiz

1. Convert the following non-linear equations to the linear form. Hence, identify Y, X, gradient and Y-intercept.

(a) $y = 5x^2 + 3x$ (b) $y = p\sqrt{x} + \frac{q}{\sqrt{x}}$ (c) $y = ax^b$

(d) $x = mxy + ny$ (e) $yp^x = q$ (f) $y(b - x) = ax$

2. The table below shows the data which relates the variables x and y by the equation $y = ax^3 + bx^2$, such that a and b are constants.

x	0.5	1.0	1.5	2.0	2.5	3.0
y	0.31	2.05	6.19	14.00	26.30	45.00

(a) Convert the non-linear equation $y = ax^3 + bx^2$ to the linear form.

(b) Plot the graph of $\frac{y}{x^2}$ against x, by using suitable scales on the x-axis and $\frac{y}{x^2}$-axis. Hence, draw the line of best fit.

(c) From the graph, find the value of a and of b.

3. The table below shows the data which relates the variables x and y by the equation $y = a^{b+x}$, such that a and b are constants.

x	1	2	3	4	5
y	2.83	5.66	11.31	22.63	45.25

(a) Convert the non-linear equation $y = a^{b+x}$ into a linear equation.

(b) Plot the graph of $\log_{10} y$ against x, by using suitable scales on the x-axis and $\log_{10} y$-axis. Hence, draw the line of best fit.

(c) From the graph, find the value of a and of b.

6.3 Application of Linear Law

Solving problems involving linear law

Example 9 MATHEMATICS APPLICATION

An experiment is carried out to study the effect of the growth of a plant on the concentration of a hormone. The readings from the experiment are recorded in the table below. The growth of the plant and the concentration of hormone are related by the equation $P = 180 + rK - sK^2$, such that r and s are constants.

Concentration of hormone per million (K)	1	3	4	6	8	10
% growth of plant (P)	181	179.7	178	168	157	140

(a) Plot the graph of $\frac{P-180}{K}$ against K, by using a scale of 2 cm to 2 units on the X-axis and 2 cm to 1 unit on the Y-axis. Hence, draw the line of best fit.
(b) From the graph, calculate the value of r and s.

Solution

1. Understanding the problem

- Identify the variables to determine the X-axis and Y-axis.
- Plot the graph by using the scales given.
- Based on the graph, find the value of r and s.

2. Planning the strategy

(a) Convert the non-linear equation to the linear form and compare with the form $Y = mX + c$, such that m is the gradient and c is the Y-intercept.
(b) Construct a new table using the new variables.
(c) Plot the graph by using the values in the new table.
(d) Find the y-intercept and gradient by refering to the graph. Hence, compare with the equation $Y = mX + c$.

3. Implementing the strategy

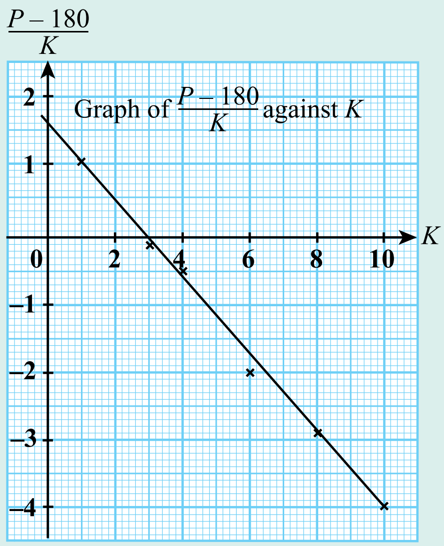
(a) $P = 180 + rK - sK^2$

$$P - 180 = rK - sK^2$$
$$\frac{P-180}{K} = \frac{rK}{K} - \frac{sK^2}{K}$$
$$\frac{P-180}{K} = r - sK$$
$$\frac{P-180}{K} = -sK + r$$

CHAPTER 6

Through comparison, $Y = \frac{P - 180}{K}$, $X = K$, $m = -s$ and $c = r$.

K	1	3	4	6	8	10
$\frac{P - 180}{K}$	1.00	−0.10	−0.50	−2.00	−2.88	−4.00



(b) Y-intercept = 1.6

$r = 1.6$

Gradient, $-s = \frac{-4 - 1.6}{10 - 0}$

$-s = -0.56$

$s = 0.56$

4. Making a conclusion

When $K = 1$,

$P = 180 + rK - sK^2$

$= 180 + (1.6)(1) - (0.56)(1)^2$

$= 181.04$

≈ 181

When $K = 3$,

$P = 180 + rK - sK^2$

$= 180 + (1.6)(3) - (0.56)(3)^2$

$= 179.76$

≈ 179.7

Self Practice 6.6

1. The table below shows the total population of a type of bacteria in a test tube. The variable x represents the number of hours and y represents the total population. Variables x and y are related by the equation $y = pq^x$, such that p and q are constants.

x (Number of hours)	2	4	6	8	10	16
y (Total population)	3.98	6.31	10.00	15.85	25.12	100.00

(a) Plot $\log_{10} y$ against x, by using suitable scales on both axes. Hence, draw the line of best fit.

(b) From the graph, find the value of

(i) p (ii) q

(c) Estimate the total population of bacteria after 5 hours.

6.3.1

CHAPTER 6

2. The table below shows the values of two variables, x and y obtained from an experiment. The variables x and y are related by the equation $xy - yb = a$, such that a and b are constants.

x	0.485	1.556	4.528	10.227	18.333	100.000
y	20.60	18.00	13.25	8.80	6.00	1.40

(a) Plot y against xy, by using suitable scales on both axes. Hence, draw the line of best fit.
(b) From the graph, find the value of a and of b.
(c) Another method of getting a straight line graph for the above non-linear equation is by plotting $\frac{1}{y}$ against x. Without drawing the second graph, calculate the gradient and the Y-intercept of the graph.

Intensive Practice 6.3

Scan the QR code or visit bit.ly/2p2uoA1 for the quiz

1. Diagram (a) and Diagram (b) show two straight line graphs which are related by the equation $y\sqrt{x} = 10$. State the value of p in the following cases.

(a)

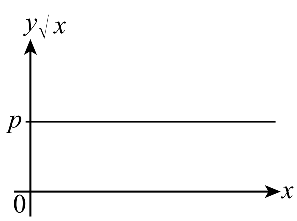


Diagram (a)

(b)

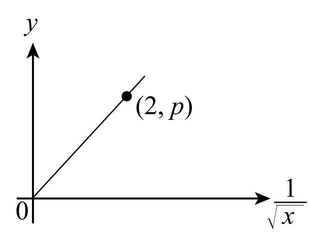


Diagram (b)

2. The table below shows the data obtained from an experiment on a pendulum such that p is the length, in cm of the pendulum and t is the period of oscillations, in seconds. One of the values of t was recorded wrongly.

Length, p (cm)	10	20	30	40	50	60
Period of oscillations, t (s)	6.3	9.0	11.0	12.6	14.1	15.0

(a) Plot the graph of t^2 against p, by using suitable scales. Hence, draw the line of best fit.
(b) Mark ⊗ on the graph, for the point which was recorded wrongly. Then, find the correct value of t.
(c) Use the graph to find the value of k if t and p are related by the equation $\sqrt{p} = \frac{t}{k}$, such that t and p are constants.

CHAPTER 6

6.3.1

3. The total production of a type of commodity, N, is related to the total number of hours, H by the equation $2N^2 - a = \frac{b}{H}$. The table below shows the corresponding value of N and of H.

H (hours)	20	40	60	80	100
N (metric tonnes)	1.225	1.162	1.140	1.135	1.127

(a) Plot the line of best fit of N^2H against H, by using suitable scales.
(b) Use the graph in (a) to find the value of a and of b.
(c) From the graph, estimate the total production if the total number of hours is 10.
(d) The manager of the company plans to produce1.1183 metric tonnes of commodity. If a worker works for 8 hours, how many workers are needed by the company?

4. The table below shows the values in an experiment involving the concentration of liquid, L unit^3, which is related to temperature, T, by the equation $L = A(3)^{\frac{b}{T}}$.

T (°C)	0.100	0.033	0.020	0.014	0.011	0.010
L (unit^3)	6.31×10^8	1.00×10^{10}	1.58×10^{11}	3.98×10^{12}	2.51×10^{13}	1.58×10^{14}

(a) Plot the line of best fit of $\log_{10} L$ against $\frac{1}{T}$, by using suitable scales.
(b) Use the graph in (a) to find the value of
(i) A,
(ii) b.
(c) Determine the temperature when the liquid is heated until its concentration achieves 21.5 unit^3.

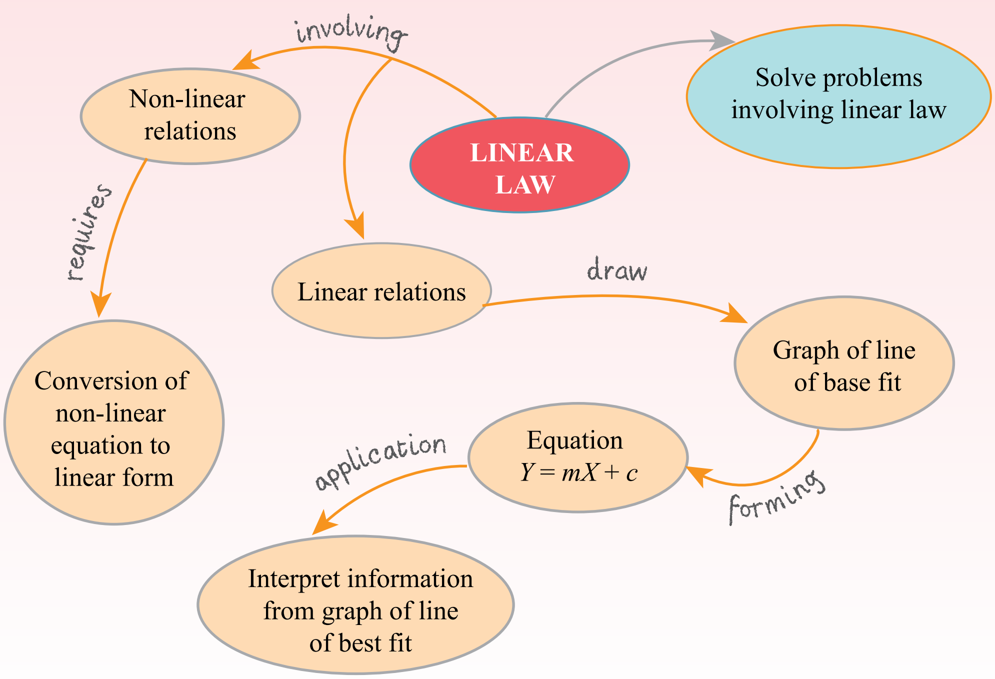
5. The table below shows the points obtained in an experiment involving two variables, u and v which are related by the relation $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$.

u	15	20	25	50	100
v	30.0	20.2	16.6	12.5	11.1

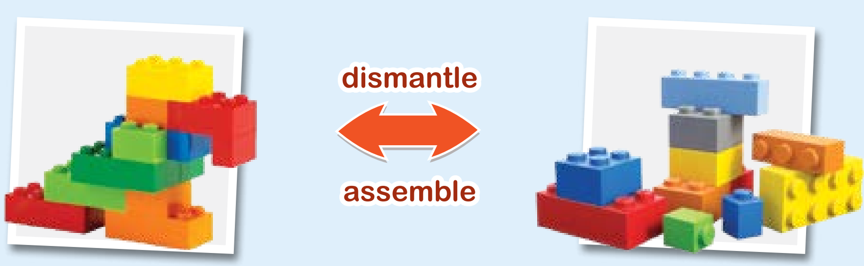
(a) Plot $\frac{1}{v}$ against $\frac{1}{u}$. Draw the line of best fit.
(b) From the graph,
(i) express v in terms of u.
(ii) determine the value of $\frac{1}{f}$ when $\frac{1}{u} = 0$, Hence, find the value of f.

CHAPTER 6

SUMMARY OF CHAPTER 6



WRITE YOUR JOURNAL



The above diagram shows the building blocks which are assembled and can be dismantled. In mathematics, there are many examples with inverses. You can convert non-linear equations into linear equations and vice versa. Can you determine the steps needed to convert linear equations into non-linear equations?

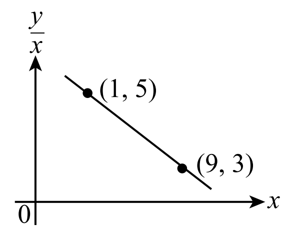
MASTERY PRACTICE

1. Express the following non-linear equations in the linear form, $Y = mX + c$, such that X and Y are variables, m and c are constants. PL2

(a) $y = 3x + \dfrac{4}{x^2}$ (b) $y = px^3 + qx^2$ (c) $y = \dfrac{p}{x} + \dfrac{q}{p}x$

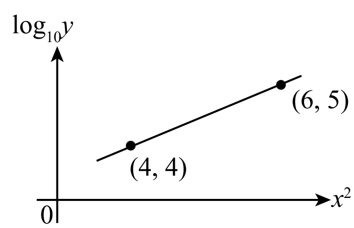
(d) $y = pk^{\sqrt{x}}$ (e) $y = pk^{x-1}$ (f) $y = \dfrac{k^{x^2}}{p}$

2. The variables x and y are related by the equation $y = px^2 + qx$, such that p and q are constants. The diagram on the right shows part of the line of best fit obtained by plotting the graph of $\dfrac{y}{x}$ against x. PL3

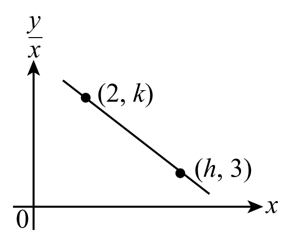


(a) Convert the equation $y = px^2 + qx$ to the linear form.
(b) Find the value of p and q.

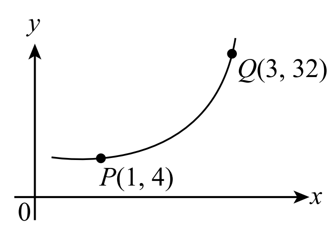
3. The variables x and y are related by the equation $y = pq^{\frac{x^2}{4}}$. The diagram on the right shows part of the line of best fit obtained by plotting $\log_{10} y$ against x^2. Find the value of p and q. PL3



4. The diagram on the right shows part of the line of best fit of $\dfrac{y}{x}$ against x. Given $y = 5x - 3x^2$, find the value of k and h. PL3



5. The diagram on the right shows part of the graph of y against x for the equation $y = ab^x$, such that a and b are constants. PL3



(a) Sketch the straight line graph of $\log_{10} y$ against x. Mark and state the coordinates of the corresponding point P and Q.
(b) Based on the graph in (a), find the value of a and b.

6. When x^2y against x are plotted, a straight line is obtained. The straight line has a gradient of 8 and it passes through point (2, 19).
(a) Determine the equation which relates x and y.
(b) Hence, find the value of y when $x = 9.4$.

CHAPTER 6

7. A study is conducted to determine the relation between the mass, m and the volume, V of a type of cooking oil. The following table shows the results of the study. PL2

V	0.5	1.0	1.5	2.0	2.5	3.0
m	0.35	0.84	1.23	1.60	2.00	2.37

Plot the graph of m against V by using a scale of 2 cm to 1 unit on both axes. Hence, draw the line of best fit.

8. Based on an experiment, the relation between the values of x and the values of y are shown in the table below. PL3

x	10	20	30	40	50	60
y	16.5	20.0	23.5	27.5	31.5	35.0

(a) Plot the graph of y against x and draw the line of best fit by using a scale of 2 cm to 10 units on the x-axis and 2 cm to 5 units on the y-axis.
(b) Hence, form the equation of the straight line.

9. The table below shows the values which relate the temperature, T of a solution after time, t in an experiment. PL4

$t(s)$	2	4	6	8	10
T(°C)	29.0	40.0	31.0	32.1	33.0

(a) Plot the graph of T against t. Hence, draw the line of best fit by using suitable scales.
(b) Mark ⊗ on the graph the point which was recorded wrongly. Then, find the correct value for T°C.
(c) From the graph, find
 (i) the initial temperature of the solution,
 (ii) the temperature of the solution after 9 seconds,
 (iii) the time taken for the solution to reach a temperature of 30.5°C.

10. The table below shows the values of two variables, x and y, obtained from an experiment. The variables x and y are related by the equation $y = st^x$, such that s and t are constants. PL3

x	1.5	3.0	4.5	6.0	7.5	9.0
y	2.51	3.24	4.37	5.75	7.76	10.00

(a) Plot the graph of $\log_{10} y$ against x, by using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 0.1 unit on the $\log_{10} y$-axis. Hence, draw the line of best fit.
(b) From the graph, find the value of
 (i) s,
 (ii) t,
 (iii) x when $y = 4$.

11. The table below shows the values of two variables, x and y, obtained from an experiment. The variables x and y are related by the equation $2y - p = \frac{q}{x}$, such that p and q are constants. PL3

x	1	2	3	4	5	6
y	5	3.5	3.1	2.7	2.6	2.5

(a) Plot the graph of xy against x, by using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 2 units on the xy-axis. Hence, draw the line of best fit.
(b) Use the graph in (a) to find the value of
  (i) p,
  (ii) q,
  (iii) y when $x = 3.5$.
(c) Calculate the value of x when $y = 50$.

Exploring MATHEMATICS

Durian is a well-known fruit in Southeast Asia. Attempts to export durians in frozen form have been carried out in order to promote durians to countries outside Southeast Asia. Manuring of plants must be carried out regularly in order to increase the production of durians. The following table shows the relation between the age and the mass of durian trees by using the recommended manuring method during the vegetative stage.

Age (year)	1	2	3	4	5
Mass (kg)	0.5	1.0	2.0	2.8	4.0

(a) Draw a dispersion diagram for the data in the table. Does the dispersion diagram show a linear relation between the age and mass of durian trees which used the manuring method?
(b) By using suitable scales, draw the line of best fit by taking the mass as a dependent variable and the age as an independent variable. Then, find the equation relating the two variables.
(c) Convert the non-linear relation to the linear form and construct a new table for the variables involved.
(d) From your graph, predict the mass of a durian tree of 7 years.

CHAPTER 7 Coordinate Geometry

What will be learnt?

- Divisor of a Line Segment
- Parallel Lines and Perpendicular Lines
- Areas of Polygons
- Equations of Loci

List of Learning Standards

bit.ly/2p7Q7qq

KEYWORDS

- Divisor of line segment — *Pembahagi tembereng garis*
- Parallel straight lines — *Garis lurus selari*
- Perpendicular straight lines — *Garis lurus serenjang*
- Gradient — *Kecerunan*
- Area of polygon — *Luas poligon*
- Equation of a locus — *Persamaan lokus*

The use of a GPS (Global Positioning System) navigation application allows us to quickly and easily locate the places that we want to go. Do you know that GPS navigation uses the idea of coordinate geometry known as World Geodetic System (WGS 84) to determine the location of places on the surface of the Earth?

Did you Know?

Ibrahim Ibn Sinan (908 – 946 AD) is a mathematician and astronomer from Harran in northern Mesopotamia. He started doing research on geometry and astronomy at the age of 15 and recorded his first research results at the age of 16. He continued Archimedes' research on area, volume and in particular, tangent to a circle.

For further information:

bit.ly/2B3TfpM

SIGNIFICANCE OF THIS CHAPTER

- In construction, coordinate geometry is used when drawing sketches of buildings.
- Astrophysicists use coordinate geometry to determine the distance between planets.
- Coordinate geometry is used in aviation to determine the angles in the flight paths of aircraft.

Scan this QR code to watch video on application of GPS.

bit.ly/2JyqSoC

7.1 Divisor of a Line Segment

A line segment is part of a straight line with two end points with specific length or distance. Any point dividing the segment in a particular ratio is known as the internal point.

Relating the position of a point that divides a line segment with the related ratio

INQUIRY 1 In groups

Aim: To explore the relationship between the position of a point on a line segment and its ratio

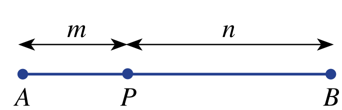
Instructions:

1. Scan the *QR* code or visit the link on the right.

bit.ly/2IvAJe2

2. Drag point P on the line segment AB to the left and right. Observe the values of m and n displayed.
3. What is the relationship between the position of point P on the line segment and the values of m and n?
4. Consider the position of point P on the line segment AB and answer the following questions.
   (a) How many parts are there between point P and point A?
   (b) How many parts are there between point P and point B?
   (c) How many parts are there between point A and point B?
   (d) What are the lengths of AP and PB in terms of AB?
   (e) Determine the ratio $AP : PB$.
   (f) What is the relationship between the position of point P on line segment AB and the ratio obtained in (e)?
5. Now, drag point P so that the ratio $m : n$ is $5 : 5 = 1 : 1$. Is the length of AP same as PB? Determine the position of point P on the line segment when the ratio $m : n$ is the same for every part.
6. Change the ratio $m : n$ and observe the position of point P. Does the position change with the change in ratio value?

From the results of Inquiry 1, point P on the line segment AB divides the line segment into two parts in the ratio $m : n$. Ratio $m : n$ means line segment AB is divided into $(m + n)$ equal parts.



The position of point P on line segment AB determines the m number of equal parts from point A to point P and the n number of equal parts from point B to point P. So, point P divides the line segment in the ratio $m : n$. Conversely, the ratio $m : n$ will determine the position of point P on the line segment AB. When ratio $m : n$ changes, the position of point P will also change. If $m = n$, then point P is the midpoint of the line segment AB. In general,

> The position of point P on a line segment AB divides the line segment in the ratio $m : n$ and vice versa.

Example 1

Given line segment PQ, a point R is on PQ. Point R is $\frac{7}{9}$ of the distance PQ from point P along the line segment PQ.

(a) Sketch this situation by using the line segment.
(b) Is point R closer to P or Q? Explain.
(c) By using the information given, determine the following ratios.
(i) $PR : PQ$, (ii) $RQ : PR$, (iii) $PR : RQ$.
(d) Hence, describe the relationship between the position of point R on line segment PQ and its ratio.

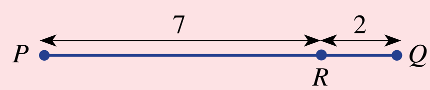
Solution

(a) 

(b) Point R is closer to Q because the position of point R is more than half of the line segment from point P.

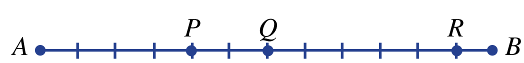
(c) (i) $PR : PQ = 7 : 9$
(ii) $RQ : PR = 2 : 7$
(iii) $PR : RQ = 7 : 2$

(d) Point R divides the line segment PQ in the ratio 7 : 2



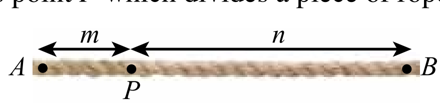
Self Practice 7.1

1. The diagram below shows a line segment AB that is divided into 12 equal parts.



P, Q and R are internal points on the line segment.
(a) Determine the position of each point in relation to its ratio.
(b) If point S is on the line segment AB in the ratio 5 : 7, mark and label the position of point S on the line segment.

2. The diagram below shows point P which divides a piece of rope AB in the ratio $m : n$.



Given $AP = 10$ cm and $AB = 35$ cm.
(a) Find the values of m and n.
(b) Describe the position of P on the rope in relation to its ratio.
(c) If the rope is placed on the x-axis of the Cartesian plane such that A is the origin and coordinates of B is $(21, 0)$, determine the coordinates of P.

CHAPTER 7

Deriving the formula for divisor of a line segment on a Cartesian plane

In Diagram 7.1, the coordinates of points A and B are (x_1, y_1) and (x_2, y_2) respectively. $P(x, y)$ is a point which divides line segment AB in the ratio $m : n$. So, $CD = x - x_1$, $DE = x_2 - x$, $PG = y - y_1$ and $BF = y_2 - y$. Since AC, PD and BE are parallel, we get:

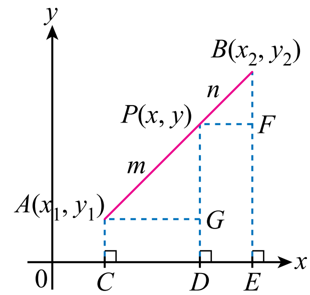


Diagram 7.1

$$\frac{CD}{DE} = \frac{AP}{PB}$$
$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$
$$n(x - x_1) = m(x_2 - x)$$
$$nx - nx_1 = mx_2 - mx$$
$$mx + nx = nx_1 + mx_2$$
$$x(m + n) = nx_1 + mx_2$$
$$x = \frac{nx_1 + mx_2}{m + n}$$

For BF, PG and AC which are also parallel, we get:

$$\frac{PG}{BF} = \frac{AP}{PB}$$
$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}$$
$$n(y - y_1) = m(y_2 - y)$$
$$ny - ny_1 = my_2 - my$$
$$my + ny = ny_1 + my_2$$
$$y(m + n) = ny_1 + my_2$$
$$y = \frac{ny_1 + my_2}{m + n}$$

Thus, the coordinates of point $P(x, y)$ which divides the line segment joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ is:

$$P(x, y) = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}\right)$$

What will happen when $m = n$? When $m = n$, P will become the midpoint of line segment AB and is represented by M.

$$M = \left(\frac{mx_1 + mx_2}{m + m}, \frac{my_1 + my_2}{m + m}\right)$$
$$= \left(\frac{m(x_1 + x_2)}{2m}, \frac{m(y_1 + y_2)}{2m}\right)$$
$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

QR

Scan the QR code below for another method of deriving formula for divisor of line segment on a Cartesian plane.

bit.ly/35ml4aS

BRAINSTORMING

Using Pythagoras Theorem, show that the distance of line segment AB represented by d in Diagram 7.1 is

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

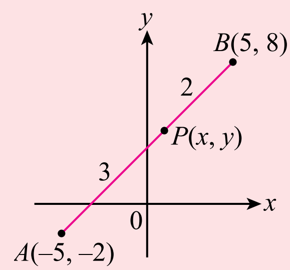
Example 2

(a) The coordinates of points A and B are $(–5, –2)$ and $(5, 8)$. If point P divides line segment AB in the ratio $3 : 2$, find the coordinates of point P.

(b) Points $A(–7, 3)$, $P(5, –3)$, B and M are on a straight line. Given P divides line segment AB in the ratio $3 : 1$ and M is the midpoint of AB. Find
  (i) the coordinates of B,
  (ii) the coordinates of M.

Solution

(a) $P(x, y)$ is the point that divides AB in the ratio $3 : 2$. So,

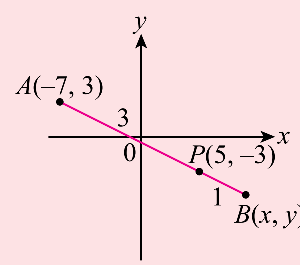


$$x\text{-coordinate of } P,\ x = \frac{2(–5) + 3(5)}{3 + 2} = \frac{–10 + 15}{5} = \frac{5}{5} = 1$$

$$y\text{-coordinate of } P,\ y = \frac{2(–2) + 3(8)}{3 + 2} = \frac{–4 + 24}{5} = \frac{20}{5} = 4$$

Thus, coordinates of point P are $(1, 4)$.

(b) (i) B is (x, y) and $P(5, –3)$ divides AB in the ratio $3 : 1$. So,

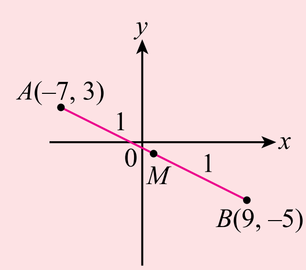


$$x\text{-coordinate of } P = 5$$
$$\frac{1(–7) + 3x}{3 + 1} = 5$$
$$3x – 7 = 20$$
$$3x = 27$$
$$= 9$$

$$y\text{-coordinate of } P = –3$$
$$\frac{1(3) + 3y}{3 + 1} = –3$$
$$3 + 3y = –12$$
$$3y = –15$$
$$y = –5$$

Thus, coordinates of point B are $(9, –5)$.

(ii)



$$\text{Midpoint of } AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{–7 + 9}{2}, \frac{3 + (–5)}{2}\right) = (1, –1)$$

Thus, coordinates of point M are $(1, –1)$.

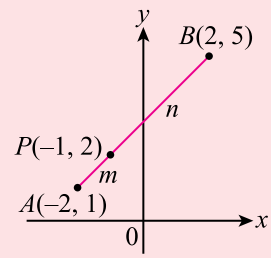
Example 3

Find the ratio $AP : PB$ such that point $P(-1, 2)$ divides the line segment joining point $A(-2, 1)$ and point $B(2, 5)$.

Solution

Let $P(-1, 2)$ divides AB in the ratio $m : n$ and the x-coordinate of P is -1.

$$\frac{n(-2) + m(2)}{m + n} = -1$$
$$2m - 2n = -m - n$$
$$3m = n$$
$$\frac{m}{n} = \frac{1}{3}$$



Thus, ratio $AP : PB$ is $1 : 3$.

Self Practice 7.2

1. Point P divides the line segment joining points A and B below in the given ratios. Find the coordinates of point P.
   (a) $A(3, 7)$, $B(-7, 2)$ in the ratio $3 : 2$.
   (b) $A(-4, -1)$, $B(2, 5)$ in the ratio $2AP : PB$.
   (c) $A(7, -3)$, $B(-3, 2)$ in the ratio $3AP : 2PB$.
2. Point $R(p, t)$ divides the line segment joining points $A(2h, h)$ and $B(2p, 3t)$ in the ratio $2 : 3$. Express p in terms of t.
3. A straight line passes through points $A(-2, -5)$ and $B(6, 7)$. Point C divides the line segment AB in the ratio $3 : 1$ while D divides AB in the ratio $1 : 1$. Find
   (a) the coordinates of C,
   (b) the coordinates of D.
4. Point P divides the line segment joining the points A and B in the ratio $AP : PB$. Find the ratio $AP : PB$ and the value of k for each of the following:
   (a) $A(1, k)$, $B(-5, 10)$ and $P(-1, 2)$
   (b) $A(1, 2)$, $B(k, 6)$ and $P(3, 4)$
   (c) $A(k, 3)$, $B(2, 8)$ and $P(6, 4)$
   (d) $A(-3, -2)$, $B(2, 8)$ and $P(-1, k)$

CHAPTER 7

Solving problems involving divisor of line segment

Example 4

MATHEMATICS APPLICATION

A spider is at position $E(-7, -5)$ on a graph paper and moves towards point $G(13, 5)$ along a straight line with uniform velocity. The spider is at point P after moving for 18 seconds and arrives at point G in 1 minute. Determine
(a) the coordinates of point P,
(b) the ratio $EQ : QG$ when the spider is at point $Q(11, 4)$ on the straight line.

Solution

1. Understanding the problem

- The original position of the spider is $E(-7, -5)$. The spider arrives at point $G(13, 5)$ in 1 minute (60 seconds).
- Find the coordinates of P after moving for 18 seconds.
- Find the ratio $EQ : QG$ when the spider is at point $Q(11, 4)$.

2. Planning the strategy

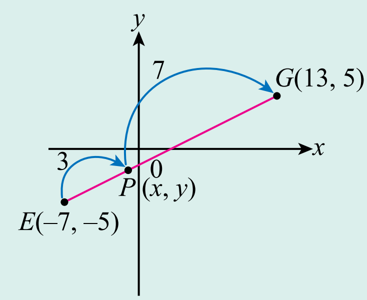
- Find the ratio $EP : PG$ first and use the formula for divisor of line segment, $P(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ to determine the coordinates of P.
- Use the formula $\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ again to determine the ratio $EQ : QG$.

3. Implementing the strategy

(a) The spider is at $P(x, y)$ after moving for 18 seconds.
The ratio $EP : EG$ is $18 : 60 = 3 : 10$, so the ratio $EP : PG = 3 : 7$.

$$P(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$$
$$= \left(\frac{7(-7) + 3(13)}{3+7}, \frac{7(-5) + 3(5)}{3+7}\right)$$
$$= \left(\frac{-10}{10}, \frac{-20}{10}\right)$$
$$= (-1, -2)$$

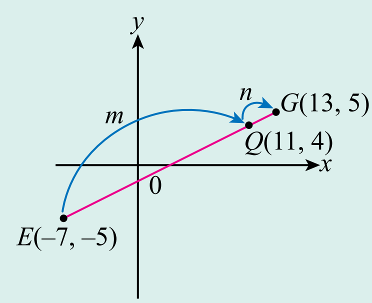
Thus, the coordinates of P are $(-1, -2)$.



(b) Let $Q(11, 4)$ divides EG in the ratio $m : n$.
The y-coordinate of Q is 4,

$$\frac{n(-5) + m(5)}{m+n} = 4$$
$$5m - 5n = 4m + 4n$$
$$m = 9n$$
$$\frac{m}{n} = \frac{9}{1}$$

Thus, the ratio $EQ : QG$ is $9 : 1$.



CHAPTER 7

4. Making a conclusion

(a) $EP = \sqrt{(-1-(-7))^2 + (-2-(-5))^2}$
$= \sqrt{6^2 + 3^2}$
$= \sqrt{45}$
$= 3\sqrt{5}$

$PG = \sqrt{(13-(-1))^2 + (5-(-2))^2}$
$= \sqrt{14^2 + 7^2}$
$= \sqrt{245}$
$= 7\sqrt{5}$

Thus, the ratio $EP : PG = 3\sqrt{5} : 7\sqrt{5} = 3 : 7$.

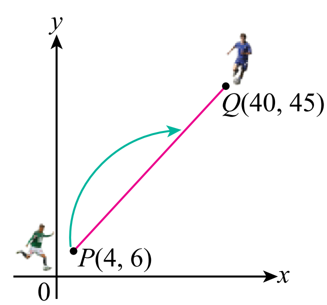
(b) $EQ = \sqrt{(11-(-7))^2 + (4-(-5))^2}$
$= \sqrt{18^2 + 9^2}$
$= \sqrt{405}$
$= 9\sqrt{5}$

$QG = \sqrt{(13-11)^2 + (5-4)^2}$
$= \sqrt{2^2 + 1^2}$
$= \sqrt{5}$

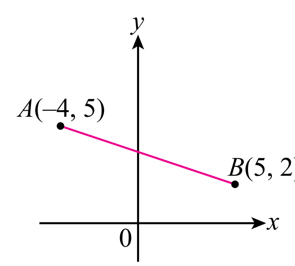
Thus, the ratio $EQ : QG = 9\sqrt{5} : \sqrt{5} = 9 : 1$.

Self Practice 7.3

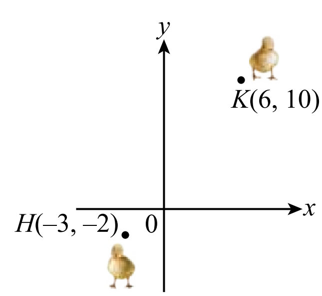
1. The diagram on the right shows the position of two football players, P and Q. The coordinates of players P and Q are (4, 6) and (40, 45) respectively. Player P wants to kick the ball to player Q but the ball lands at $\frac{2}{3}$ the distance of the straight line towards player Q from player P. Determine the coordinates of the ball when it touches the surface of the field.



2. The diagram on the right shows the plan of a straight highway between two towns, A and B on the Cartesian plane. An engineer wants to build two rest houses between the two towns such that the two rest houses divide the road into three parts of equal distance. Determine the coordinates of the two rest houses.



3. The diagram on the right shows the position of two ducklings, H and K on the Cartesian plane. Given the coordinates of duckling H are (–3, –2) and the coordinates of duckling K are (6, 10). The two ducklings walk towards each other with different velocities and meet at point L. The velocity of duckling H is twice the velocity of duckling K.
   (a) State the ratio $HL : LK$.
   (b) Find the distance of duckling K from its original position when duckling K meets duckling H.



Intensive Practice 7.1

Scan QR code or visit bit.ly/2pT22Zk for the quiz

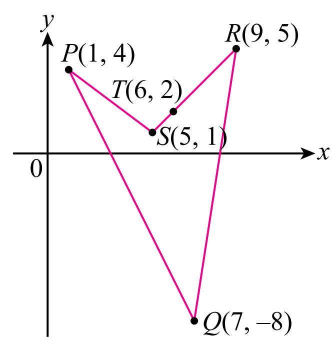
1. A straight line passes through $P(2, 8)$ and $Q(7, 3)$. Point R divides line segment PQ such that $PR = 4QR$. Find the coordinates of point R.

2. If the point $R(6, 3)$ divides the line segment from $P(4, 5)$ to $Q(x, y)$ in the ratio 2 : 5, find
(a) the coordinates of Q,
(b) the coordinates of the midpoint of PQ.

3. Point $C(1, 4)$ divides the straight line joining points $A(–3, 6)$ and $B(h, k)$ in the ratio 2 : 3. Find the value of h and k.

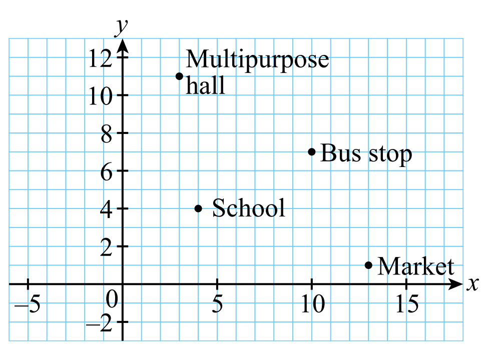
4. Points $A(4r, r)$, $B(e, f)$ and $C(3e, 4f)$ are on a straight line. B divides the straight line AC in the ratio 3 : 4. Express e in terms of f.

5. The diagram on the right shows a quadrilateral $PQRS$ with vertices $P(1, 4)$, $Q(7, –8)$, $R(9, 5)$ and $S(5, 1)$. Point $T(6, 2)$ is on the straight line RS. Find
(a) the coordinates of point U which divides the side PQ in the ratio 2 : 1,
(b) the coordinates of the midpoint of side QR,
(c) the ratio $RT : TS$,
(d) the length of side PS.



6. Point $P(k, 2)$ divides the straight line joining the points $A(–2, 1)$ and $B(2, 5)$ in the ratio $m : n$. Find
(a) the ratio $m : n$,
(b) the value of k.

7. The diagram below shows the position of the multipurpose hall, school, market and bus stop on a Cartesian plane. Haziq's house is at the midpoint of P_1P_2 such that P_1 divides the line segment from the multipurpose hall to the market in the ratio 4 : 1, while P_2 divides the line segment from the school to the bus stop in the ratio 1 : 2.



Determine the position of Haziq's house.

CHAPTER 7

7.2 Parallel Lines and Perpendicular Lines

It is easy to find parallel lines and perpendicular lines around us. The floats separating the lanes in a swimming pool and the support structures used in construction are some of the examples of parallel and perpendicular lines. What are some other examples of parallel and perpendicular lines around us?

Making and verifying conjectures about gradient of parallel and perpendicular lines

INQUIRY 2 In groups | 21st Century Learning

Aim: To make and verify conjectures about the relationship between the gradient of two parallel lines and the gradient of two perpendicular lines

Instructions:

1. Form two groups and each group will choose one activity.

ACTIVITY 1

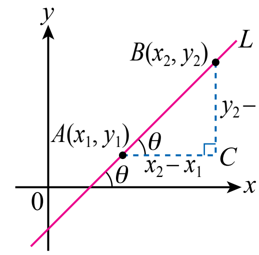
1. Using *GeoGebra* software, draw straight lines L_1 and L_2 that are parallel to each other on the Cartesian plane.
2. Record the gradient of straight lines L_1 and L_2.
3. Drag straight lines L_1 or L_2 and observe the changes on the gradient of L_1 and L_2.
4. What can you say about the relationship between the gradient of straight lines L_1 and L_2?
5. Measure the angles formed between the lines L_1 and L_2 and the positive x-axis. What can you observe about the two angles? Explain.
6. With your group members, verify the relationship you obtained in step **4** with the results you obtained in step **5**.

ACTIVITY 2

1. Using *GeoGebra* software, draw straight lines L_1 and L_2 that are perpendicular to each other on the Cartesian plane.
2. Record the gradient of L_1 and L_2 and determine the product of gradient of L_1 and L_2.
3. Drag straight lines L_1 or L_2 and observe the change in the gradient of L_1 and L_2 as well as the product of their gradient.
4. What can you say about the relationship between the gradient of L_1 and L_2?
5. Measure θ_1 and θ_2, the angles formed between lines L_1 and L_2 respectively towards positive x-axis. Hence, determine the product of tan θ_1 and tan θ_2.
6. What is the relationship between tan θ_1 and tan θ_2? Explain.
7. With your group members, verify the relationship you obtained in step **4** with the results you obtained in step **6**.

2. Appoint a representative from each group to present the results obtained in front of the class.

We have learnt that the gradient, m of a straight line L which passes point $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the formula:



$$\text{Gradient, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

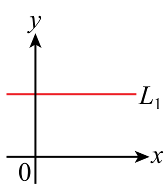
As shown in the diagram on the right, in ΔABC,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{BC}{AC}$$
$$m = \tan \theta$$

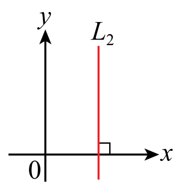
So, the definition of gradient, m of a straight line is:

> $m = \tan \theta$
> with θ being the angle formed between a straight line and the positive x-axis and $0° \leqslant \theta < 180°$.

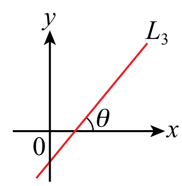
The following diagrams show the gradient of a straight line L changing when θ increases from 0° to 180°.



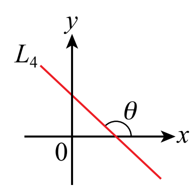
$\theta = 0°$,
$\tan \theta = \tan 0°$
$m_{L_1} = 0$



$\theta = 90°$,
$\tan \theta = \tan 90°$
m_{L_2} not defined

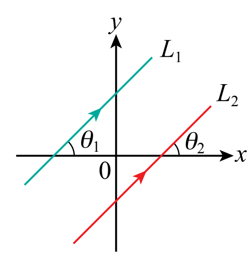


$0° < \theta < 90°$,
$\tan \theta > 0$
$m_{L_3} > 0$



$90° < \theta < 180°$,
$\tan \theta < 0$
$m_{L_4} < 0$

Thus, from activity 1 in Inquiry 2, let m_1 and m_2 be the gradient of straight lines L_1 and L_2 respectively. If lines L_1 and L_2 are parallel, then



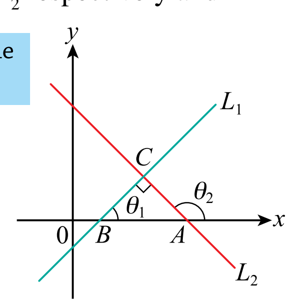
$\theta_1 = \theta_2$ ← Corresponding angles, lines //
$\tan \theta_1 = \tan \theta_2$
that is, $m_1 = m_2$

Conversely, if $m_1 = m_2$, we can find that $\theta_1 = \theta_2$ and L_1 is parallel to L_2.

> Two straight lines, L_1 and L_2 are parallel to each other if and only if $m_1 = m_2$.

From the results of activity 2 in Inquiry 2, let m_1 and m_2 be the gradient of straight lines L_1 and L_2 respectively and $\theta_1 \neq 0$.

$\theta_2 = 90° + \theta_1$ ← Exterior angle of Δ
$\tan \theta_2 = \tan (90° + \theta_1)$
$\tan \theta_2 = -\dfrac{1}{\tan \theta_1}$
$\tan \theta_1 \tan \theta_2 = -1$
that is, $m_1 m_2 = -1$



Smart TIPS

In ΔABC,
$\tan \theta_1 = \dfrac{AC}{BC}$
$\dfrac{BC}{AC} = \dfrac{1}{\tan \theta_1}$
So, $\tan \theta_2 = -\dfrac{BC}{AC}$
$= -\dfrac{1}{\tan \theta_1}$

CHAPTER 7

7.2.1

Conversely, if $m_1 m_2 = -1$ we can find that $\theta_2 = 90° + \theta_1$ and L_1 is perpendicular to L_2.

Two straight lines, L_1 and L_2 are perpendicular to each other if and only if $m_1 m_2 = -1$.

Example 5

(a) Show whether the straight lines $6x + 9y = 7$ and $\frac{x}{3} + \frac{y}{2} = 1$ are parallel.

(b) The straight line $y = 4 - \frac{k}{3}x$, such that k is a constant is parallel to the straight line $2x + 3y = 9$. Find the value of k.

Solution

(a) Write equation $6x + 9y = 7$ in gradient form.

$$6x + 9y = 7$$
$$9y = -6x + 7$$
$$y = -\frac{2}{3}x + \frac{7}{9}$$

← Arrange in gradient form, $y = mx + c$

Gradient, $m_1 = -\frac{2}{3}$

For straight line equation $\frac{x}{3} + \frac{y}{2} = 1$, ← Straight line in intercept form

Gradient, $m_2 = -\frac{b}{a}$

$= -\frac{2}{3}$

Since both straight lines have the same gradient, they are parallel.

(b) $y = 4 - \frac{k}{3}x$

$y = -\frac{k}{3}x + 4$

Gradient, $m_1 = -\frac{k}{3}$

$$2x + 3y = 9$$
$$3y = -2x + 9$$
$$y = -\frac{2}{3}x + 3$$

Gradient, $m_2 = -\frac{2}{3}$ ← Gradient form, $y = mx + c$

Since both of the straight lines are parallel,

$$m_1 = m_2$$
$$-\frac{k}{3} = -\frac{2}{3}$$
$$k = 2$$

MATHEMATICS POCKET

- **Gradient form**
  $y = mx + c$, where m is gradient and c is y-intercept.
- **Intercept form**
  $\frac{x}{a} + \frac{x}{b} = 1$, where a and b are the x-intercept and y-intercept respectively, and the gradient is $-\frac{b}{a}$.

CHAPTER 7

Example 6

(a) Determine whether straight lines $y - 3x = 5$ and $3y + x - 12 = 0$ are perpendicular.
(b) The vertices of a triangle ABC are $A(0, -5)$, $B(2, 1)$ and $C(-7, k)$, such that k is a constant. Find the value of k if $\angle ABC = 90°$.

Solution

(a) Write both equations in the gradient form to find their gradients.

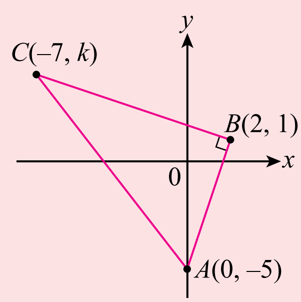
$$y - 3x = 5$$
$$y = 3x + 5$$
Gradient, $m_1 = 3$
$$3y + x - 12 = 0$$
$$3y = -x + 12$$
$$y = -\frac{1}{3}x + 4$$
Gradient, $m_2 = -\frac{1}{3}$

It is found that, $m_1m_2 = 3\left(-\frac{1}{3}\right)$
$= -1$

Thus, straight lines $y - 3x = 5$ and $3y + x - 12 = 0$ are perpendicular to each other.

(b) Since $\angle ABC = 90°$,

$$m_{AB}m_{BC} = -1$$
$$\left(\frac{1-(-5)}{2-0}\right)\left(\frac{k-1}{-7-2}\right) = -1$$
$$3\left(\frac{k-1}{-9}\right) = -1$$
$$k - 1 = 3$$
$$k = 4$$



Mind Challenge

Can Pythagoras Theorem be used to verify the answer in Example 6(b)?

Self Practice 7.4

1. Determine whether the following pairs of straight line are parallel or perpendicular to each other.

(a) $2x + 3y = 9$ and $4x + 6y = 0$
(b) $y = \frac{3}{4}x - 5$ and $4y - 3x = 12$
(c) $x - 2y = 6$ and $2x + y = 5$
(d) $2x + 3y = 9$ and $2y = 3x + 10$

2. The following pairs of straight lines are parallel, such that p is a constant. Find the value of p.

(a) $2y = 10 - x$ and $y = 3px - 1$
(b) $\frac{x}{3} - \frac{y}{6} = 1$ and $py = 4x - 6$

3. The following pairs of straight lines are perpendicular to each other. Find the value of constant k.

(a) $3x + 5y = 15$ and $5x - ky = 2$
(b) $\frac{x}{3} + \frac{y}{9} = 1$ and $ky = 2x - 7$

4. The vertices of a triangle ABC are $A(1, 1)$, $B(-1, 4)$, and $C(5, a)$. Find the value of constant a if AB is perpendicular to BC.

7.2.1

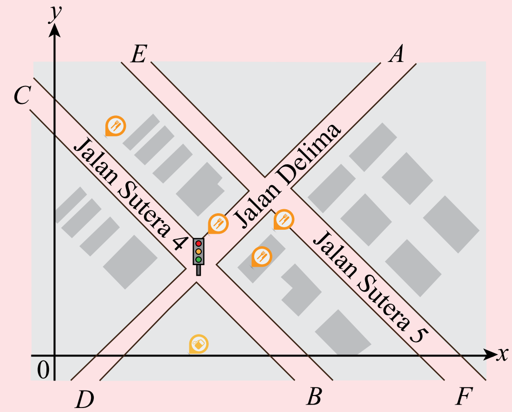
CHAPTER 7

Solving problems involving equations of parallel and perpendicular lines

Example 7 MATHEMATICS APPLICATION

The diagram on the right shows the position of roads AD, BC and EF drawn on a Cartesian plane. AD and BC are perpendicular to each other and they meet at a traffic light junction, while BC and EF are parallel to each other. Given the coordinates of A are (18, 16) and F(20, –1) while the equation of road BC is $5y + 4x = 70$, find

(a) the equation of road EF,
(b) the equation of road AD,
(c) the coordinates of the traffic light.



Solution

1. Understanding the problem

- Road AD and BC are perpendicular.
- Road BC and EF are parallel.
- Coordinates of point A are (18, 16), F are (20, –1) and the equation of road BC is $5y + 4x = 70$.
- Find the equation of roads EF and AD as well as the coordinates of traffic lights which are at the junction of roads AD and BC.

2. Planning the strategy

- Write the equation $5y + 4x = 70$ in the gradient form to determine its gradient, m_1.
- Use $m_1 = m_2$ to find the gradient of road EF.
- Use formula $m_1m_2 = -1$ to find the gradient of road AD.
- Use formula $y - y_1 = m(x - x_1)$ to find the equation of roads EF and AD.
- Solve equation $5y + 4x = 70$ and equation AD simultaneously to find the coordinates of the traffic lights.

3. Implementing the strategy

(a)
$$5y + 4x = 70$$
$$5y = -4x + 70$$
$$y = -\frac{4}{5}x + 14$$

Gradient, $m_1 = -\frac{4}{5}$, thus gradient EF which is parallel to BC is $-\frac{4}{5}$.

Equation of road EF that passes through point F(20, –1) is

$$y - (-1) = -\frac{4}{5}(x - 20)$$
$$5y + 5 = -4x + 80$$
$$5y + 4x = 75$$

CHAPTER 7

7.2.2

(b) Gradient, $m_1 = -\frac{4}{5}$, thus gradient of road AD, m_2 which is perpendicular is

$$-\frac{4}{5}m_2 = -1$$
$$m_2 = \frac{5}{4}$$

Equation of road AD which passes through point $A(18, 16)$ is

$$y - 16 = \frac{5}{4}(x - 18)$$
$$4y - 64 = 5x - 90$$
$$4y - 5x = -26$$

(c) Equation of BC: $5y + 4x = 70$ ⋯ ①
Equation of AD: $4y - 5x = -26$ ⋯ ②
① × ⑤: $25y + 20x = 350$ ⋯ ③
② × ④: $16y - 20x = -104$ ⋯ ④
③ + ④: $41y = 246$
$y = 6$

Substitute $y = 6$ into (1).
$5(6) + 4x = 70$
$30 + 4x = 70$
$4x = 40$
$x = 10$

Thus, the coordinates of traffic lights are (10, 6).

4. Making a conclusion

Substitute point $F(20, -1)$ into equation $5y + 4x = 75$.
Left side $= 5(-1) + 4(20)$
$= 75$
$=$ right side

Thus, $5y + 4x = 75$ is the equation of road EF.

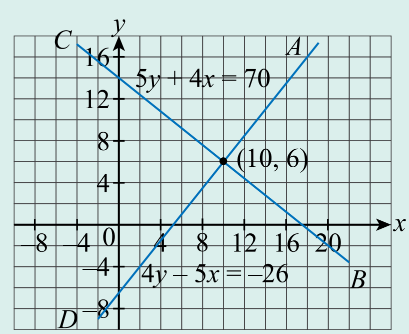
Substitute point $A(18, 16)$ into equation $4y - 5x = -26$.
Left side $= 4(16) - 5(18)$
$= -26$
$=$ right side

Thus, $4y - 5x = -26$ is the equation of road AD.

From the graph on the right, the coordinates of traffic lights are (10, 6).

FLASHBACK

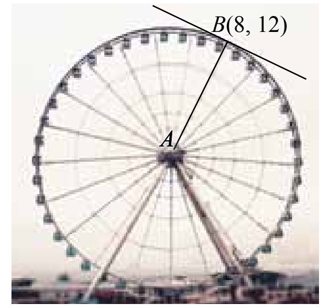
Equation of straight line with gradient m and passing through point (x_1, y_1) is $y - y_1 = m(x - x_1)$



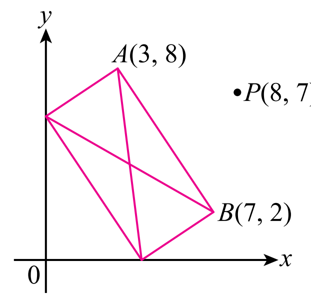
CHAPTER 7

Self Practice 7.5

1. In the diagram on the right, radius AB of the Ferris wheel is perpendicular to the tangent of the circle at point $B(8, 12)$. The equation of the tangent to the circle at point B is given as $3x + 2y = 48$. Find the equation of radius AB of the Ferris wheel.



2. The diagram on the right shows the plan of a rectangular-shaped hut drawn on a Cartesian plane. A pipe of the shortest length will be connected from the main pipe at $P(8, 7)$ to the hut. Find
   (a) the coordinates of point of connection of the pipe at the hut,
   (b) the length of trench that should be dug in order to bury the pipe to the hut.



Intensive Practice 7.2

Scan the QR code or visit bit.ly/2Vtz9P4 for the quiz

1. For each of the following, determine whether the lines AB and CD are parallel or perpendicular to each other.
   (a) $A(6, 2)$, $B(3, 4)$, $C(3, -1)$, $D(-3, 3)$
   (b) $A(4, -3)$, $B(-3, 4)$, $C(1, 4)$, $D(-2, 1)$

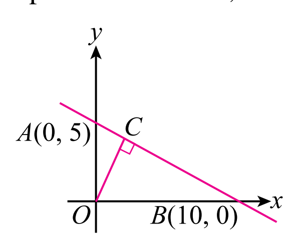
2. Given $A(1, 2)$, $B(6, 8)$ and $C(12, k)$ are the vertices of a triangle, such that $\angle ABC = 90°$, find the value of k.

3. Given $P(7, 3)$, $Q(2, 2)$ and $R(-1, 4)$. Find
   (a) the equation of straight line that passes through point P and is parallel to QR,
   (b) the equation of straight line that passes through point R and is perpendicular to QR.
   Then, find the coordinates of S such that both lines intersect.

4. The coordinates of three points are $P(-1, -6)$, $Q(3, -12)$ and $R(e, 6)$. Find the value of constant e if
   (a) P, Q and R are collinear,
   (b) PQ is perpendicular to PR.

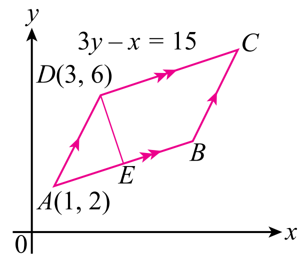
5. Given four points, $P(-6, 1)$, $Q(1, -2)$, $R(0, 5)$ and $S(-3, h)$. If PQ is perpendicular to RS, find the value of constant h.

6. In the diagram on the right, OC is perpendicular from origin O to straight line AB, such that point A is $(0, 5)$ and point B is $(10, 0)$. Find
   (a) the equation of straight line AB and OC,
   (b) the coordinates of C and distance OC.

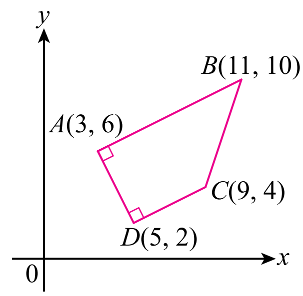


CHAPTER 7

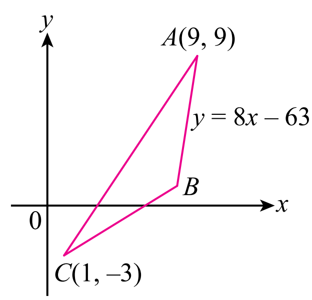
7. The diagram on the right shows a parallelogram $ABCD$. Points A and D are $(1, 2)$ and $(3, 6)$ respectively. Equation of the straight line DC is $3y - x = 15$. DE is the perpendicular bisector of AB. Find
(a) the equation for AB and DE,
(b) the coordinates of E and B.



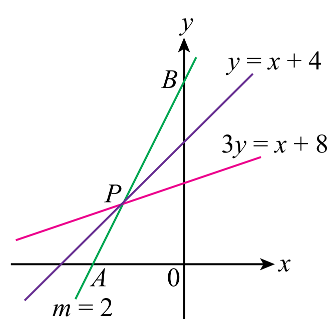
8. The diagram on the right shows a trapezium $ABCD$. The coordinates of A, B, C and D are $A(3, 6)$, $B(11, 10)$, $C(9, 4)$ and $D(5, 2)$ respectively.
(a) Determine the pairs of parallel and perpendicular lines.
(b) Find the equation of straight line AB.
(c) A straight line passes through point C and is perpendicular to AB. Find the equation of the straight line. Show that the straight line passes through the midpoint of AB.



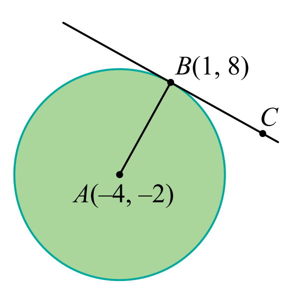
9. The diagram on the right shows a triangle ABC, such that $A(9, 9)$ and $C(1, –3)$. Point B lies on the perpendicular bisector of AC and the equation of straight line AB is $y = 8x – 63$.
(a) Find
 (i) the equation of perpendicular bisector of AC,
 (ii) the coordinates of B.
(b) Point D lies on the diagram such that $ABCD$ is a rhombus.
 (i) Find the coordinates of D.
 (ii) Show that $AC = 2BD$.



10. In the diagram on the right, two straight lines, $y = x + 4$ and $3y = x + 8$ intersect at point P. The straight line that passes through point P with gradient 2 meets the x-axis and y-axis at points A and B respectively. Show that
(a) the coordinates of P are $(–2, 2)$,
(b) the equation of straight line that passes through point P and is perpendicular to straight line AB is $2y + x = 2$,
(c) the coordinates of A are $(–3, 0)$ and the coordinates of B are $(0, 6)$,
(d) the ratio of $\dfrac{AP}{PB}$ is $\dfrac{1}{2}$.



11. In the diagram on the right, BC is a tangent to the circle with centre $A(–4, –2)$ at point $B(1, 8)$. Find the equation of tangent BC.



7.3 Areas of Polygons

Deriving formula of area of triangles

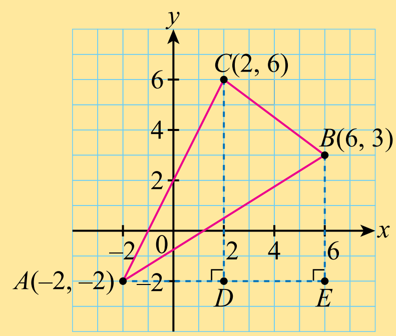
We can use formula to find the area of a polygon on the Cartesian plane if the vertices are known. Follow the exploration below to derive the formula of area of a triangle when the coordinates of each vertex are known.

INQUIRY 3 In groups 21st Century Learning

Aim: To determine area of triangle when coordinates of each vertex are known

Instructions:

1. Using *GeoGebra* software, draw a triangle with vertices *A*, *B*, and *C*.
2. Draw dashed lines as shown in the diagram on the right.
3. Using the instruction menu in the software,
   (a) find the length of *AD*, *DE*, *BE* and *CD*.
   (b) find the area of Δ*ACD*, trapezium *BCDE* and Δ*ABE*.
   (c) determine the area of Δ*ABC* by using the values obtained in (b).
4. Discuss with your group members the way to obtain area of the triangle.
5. Are there any other ways to obtain the area of triangle *ABC*?



From the results of Inquiry 3, we can make a generalisation about the method to obtain area of a triangle by using the following formula.

Diagram 7.2 shows a triangle *ABC*, with the position of $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ arranged in order.

Area of Δ*ABC*

= area of Δ*ACD* + area of trapezium *BCDE* – area of Δ*ABE*

$$= \left(\frac{1}{2} \times AD \times CD\right) + \left(\frac{1}{2} \times DE \times (BE + CD)\right) - \left(\frac{1}{2} \times AE \times BE\right)$$

$$= \frac{1}{2}(x_3 - x_1)(y_3 - y_1) + \frac{1}{2}(x_2 - x_3)[(y_2 - y_1) + (y_3 - y_1)]$$

$$- \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

$$= \frac{1}{2}(x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1 + x_2y_2 - x_2y_1 + x_2y_3 - x_2y_1 - x_3y_2$$

$$+ x_3y_1 - x_3y_3 + x_3y_1 - x_2y_2 + x_2y_1 + x_1y_2 - x_1y_1)$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

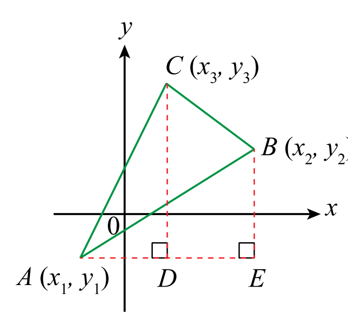


Diagram 7.2

This area formula can be written as:

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

with the sum of all the products in the ↘ direction is given the positive sign, that is $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1)$ and the sum of all the products in the ↗ direction is given the negative sign, that is $\frac{1}{2}(-x_2y_1 - x_3y_2 - x_1y_3)$.

Thus, formula for area of a triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ arranged in order can be written as:

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\right|$$

$$= \frac{1}{2}\left|(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right|$$

Determining the area of triangles by using the formula

Example 8

Find the area of ΔABC with vertices $A(-4, -6)$, $B(5, 3)$ and $C(2, 8)$.

Solution

If the coordinates are arranged anticlockwise,

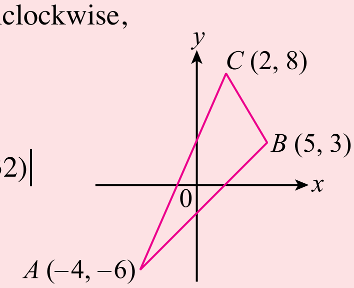
area of ΔABC

$$= \frac{1}{2}\begin{vmatrix} -4 & 5 & 2 & -4 \\ -6 & 3 & 8 & -6 \end{vmatrix}$$

$$= \frac{1}{2}\left|(-12 + 40 - 12) - (-30 + 6 - 32)\right|$$

$$= \frac{1}{2}\left|72\right|$$

$= 36$ units^2



If the coordinates are arranged clockwise,

$$\text{area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} -4 & 2 & 5 & -4 \\ -6 & 8 & 3 & -6 \end{vmatrix}$$

$$= \frac{1}{2}\left|(-32 + 6 - 30) - (-12 + 40 - 12)\right|$$

$$= \frac{1}{2}\left|-72\right| \quad \leftarrow \text{Take absolute value}$$

$$= \frac{1}{2}(72)$$

$= 36$ units^2

7.3.1 7.3.2

MATHEMATICS POCKET

The formula on the left is known as the *shoelace* algorithm which is used only when coordinates of vertices are arranged anticlockwise. If the coordinates are arranged clockwise, the answer obtained would have negative sign. In this case, the absolute value needs to be used so that the value of area is positive. This formula can be initiated by choosing any one vertex.

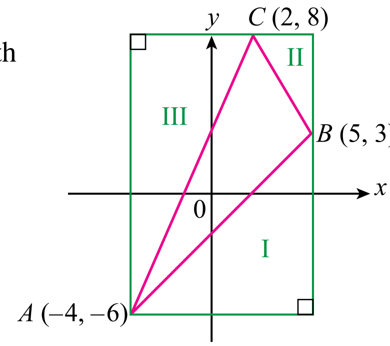
Smart TIPS

Some of the quantities we encounter in daily life have magnitude only, such as temperature, mass, distance, area and volume which do not involve direction. For example, area of ΔABC is 36 units^2. Therefore, 36 units^2 is the size or magnitude for the area of ΔABC. Such quantities are known as scalar quantities.

MATHEMATICS POCKET

The *shoelace* formula is used to find the area of a polygon when the coordinates of each vertex is known. The two vertical lines in this formula are known as the absolute value which functions to make sure that the measurement area is always positive. Important note: The measurement area of polygon accept positive value only.

Besides the *shoelace* algorithm, the box method can also be used to find the area of the triangle in Example 8.



Step 1 Draw a rectangle that touches every vertex of the triangle. Mark the triangles formed in the box with I, II and III as shown in the diagram on the right.

Step 2 Find the area of the rectangle by multiplying its length by width.

Area of rectangle $= 9 \times 14$
$= 126$ units^2

Step 3 Find the area of triangles I, II and III in the rectangle.

Area of triangle I $= \frac{1}{2} \times 9 \times 9 = 40\frac{1}{2}$ units^2

Area of triangle II $= \frac{1}{2} \times 3 \times 5 = 7\frac{1}{2}$ units^2

Area of triangle III $= \frac{1}{2} \times 6 \times 14 = 42$ units^2

Step 4 Subtract each of the area of triangles obtained in step 3 from the area of rectangle to determine the area of ΔABC.

Area of $\Delta ABC = 126 - 40\frac{1}{2} - 7\frac{1}{2} - 42$
$= 36$ units^2

BRAINSTORMING

What can you say about the three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ if the area of

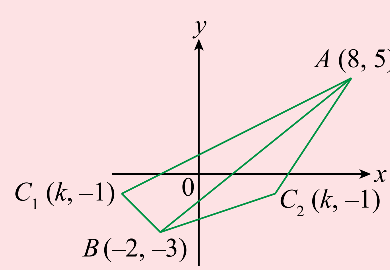
$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = 0$?

Example 9

The coordinates of the vertices of a triangle ABC are $A(8, 5)$, $B(-2, -3)$ and $C(k, -1)$. If the area of triangle ABC is 18 units^2, find the possible values of k.

Solution

Since the order of the vertices of triangle ABC is not known, we need to take the absolute value for the expression for area of triangle ABC.



$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 8 & -2 & k & 8 \\ 5 & -3 & -1 & 5 \end{vmatrix}$$

$$\pm 18 = \frac{1}{2}\left|(-24 + 2 + 5k) - (-10 - 3k - 8)\right|$$

$$\pm 18 = \frac{1}{2}(8k - 4)$$

$\frac{1}{2}(8k - 4) = -18$ or $\frac{1}{2}(8k - 4) = 18$

$8k - 4 = -36$ | $8k - 4 = 36$
$8k = -32$ | $8k = 40$
$k = -4$ | $k = 5$

Thus, the possible values of k are -4 and 5.

Self Practice 7.6

1. Find the area of the triangles with the following vertices given.
   (a) (5, 10), (2, 1), (8, 3)
   (b) (3, 1), (6, 4), (–4, 2)
   (c) (–4, –3), (5, 1), (2, 6)

2. Vertices P and Q are (3, 4) and (1, –2), respectively, and vertex R is on the x-axis. Find the coordinates of R such that area of ΔPQR is 10 units^2.

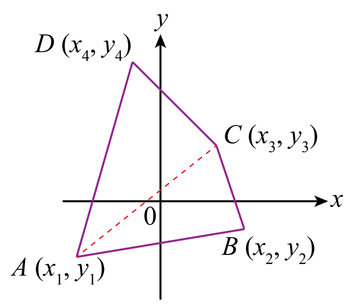
3. Show that points (8, 4), (2, 1) and (–2, –1) are collinear.

4. Points $E(–2, –1)$, $F(2, p)$ and $G(10, 5)$ are collinear. Find the value of p.

5. The vertices and area of ΔABC are given below. Find the possible values of k.
   (a) $A(-4, -1)$, $B(5, 3)$, $C(-1, k)$; area of ΔABC = 15 units^2
   (b) $A(5, k)$, $B(3, 7)$, $C(1, 3)$; area of ΔABC = 10 units^2
   (c) $A(1, -2)$, $B(k, 6)$, $C(1, 2)$; area of ΔABC = 12 units^2
   (d) $A(3, 0)$, $B(4, k)$, $C(1, 4)$; area of ΔABC = 5 units^2

Determining the area of quadrilaterals by using the formula

Consider quadrilateral $ABCD$ in the diagram on the right, with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ arranged in order.



Area of quadrilateral $ABCD$
= area of ΔABC + area of ΔACD

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_4 & x_1 \\ y_1 & y_3 & y_4 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left|(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right|$$
$$+ \frac{1}{2}\left|(x_1y_3 + x_3y_4 + x_4y_1) - (x_3y_1 + x_4y_3 + x_1y_4)\right|$$

$$= \frac{1}{2}\left|(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\right|$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$

From the expansion of the above expression, we found that the formula obtained is similar to the formula for area of triangle.

In general, the area of quadrilateral $ABCD$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ arranged in order can be written as:

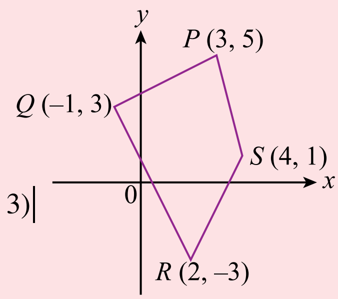
$$\text{Area of quadrilateral } ABCD = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$
$$= \frac{1}{2}\left|(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\right|$$

CHAPTER 7

Example 10

Find the area of quadrilateral $PQRS$ with vertices $P(3, 5)$, $Q(–1, 3)$, $R(2, –3)$ and $S(4, 1)$.

Solution



Arrange the vertices in order:

$$\text{Area of quadrilateral } PQRS = \frac{1}{2}\begin{vmatrix} 3 & -1 & 2 & 4 & 3 \\ 5 & 3 & -3 & 1 & 5 \end{vmatrix}$$

$$= \frac{1}{2}\left|(9 + 3 + 2 + 20) - (-5 + 6 - 12 + 3)\right|$$

$$= \frac{1}{2}\left|42\right|$$

$$= 21 \text{ units}^2$$

Self Practice 7.7

1. Find the area of the quadrilaterals with the following vertices given.
   (a) (1, 7), (–5, 6), (–2, –4) and (2, –3)
   (b) (2, 9), (–6, 4), (–1, –3) and (8, 1)
   (c) (0, 2), (–6, –2), (–3, –5) and (–1, –3)
   (d) (3, 4), (–2, 0), (2, –4) and (5, 1)

2. The vertices of a quadrilateral $ABCD$ arranged in order are $A(k, 6)$, $B(–2, 1)$, $C(4, 5)$ and $D(2, 8)$. If area of the quadrilateral $ABCD$ is 30 unit², find the value of k.

Making generalisation about the formula of area of polygons

The idea in finding the area of triangle can be used to prove that the area of a polygon with n sides and vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$, …, $N(x_n, y_n)$ is as follows.

Area of polygon

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \ldots x_n & x_1 \\ y_1 & y_2 & y_3 & y_4 \ldots y_n & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left|(\text{sum of all } \searrow \text{ products}) - (\text{sum of all } \nearrow \text{ products})\right|$$

with vertices $A, B, C, D, \ldots, N$ arranged in order.

In general, if the vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$, …, $N(x_n, y_n)$ of a polygon with n sides are arranged in order, then :

$$\text{Area of polygon} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & \ldots & x_n & x_1 \\ y_1 & y_2 & \ldots & y_n & y_1 \end{vmatrix}$$

BRAINSTORMING

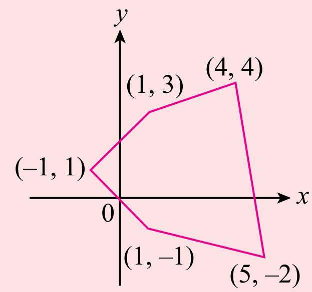
Surf the Internet to explore convex polygon and concave polygon. Discuss with your friends whether the formula for area of polygon can be used for concave polygon.

Example 11

Find the area of a pentagon with vertices (5, –2), (1, –1), (–1, 1), (1, 3) and (4, 4).

Solution

By plotting vertices of the pentagon as in the diagram on the right, the vertices arranged in order would be (4, 4), (1, 3), (–1, 1), (1, –1) and (5, –2).



$$\text{Area of pentagon} = \frac{1}{2}\begin{vmatrix} 4 & 1 & -1 & 1 & 5 & 4 \\ 4 & 3 & 1 & -1 & -2 & 4 \end{vmatrix}$$
$$= \frac{1}{2}\left|(12 + 1 + 1 - 2 + 20) - (4 - 3 + 1 - 5 - 8)\right|$$
$$= \frac{1}{2}\left|43\right|$$
$$= 21\frac{1}{2} \text{ units}^2$$

Self Practice 7.8

1. A pentagon *ABCDE* has vertices *A*(–2, –5), *B*(3, 2), *C*(2, 8), *D*(0, 9) and *E*(–3, 1). Find the area of pentagon *ABCDE*.

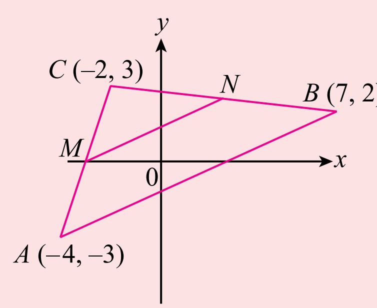
2. The vertices of a hexagon are (0, –1), (–3, –1), (–4, 2), (–2, 6), (1, 5) and (2, 1). Find the area of the hexagon.

Solving problems involving areas of polygons

Example 12

The vertices of a triangle *ABC* are *A*(–4, –3), *B*(7, 2) and *C*(–2, 3). *M* and *N* are the midpoints of the sides *AC* and *BC* respectively. Find
(a) the coordinates of *M* and *N*,
(b) the ratio of area of triangle *CMN* to area of quadrilateral *ABNM*.

Solution



(a) Coordinates of $M = \left(\frac{-4 + (-2)}{2}, \frac{-3 + 3}{2}\right)$
$= (-3, 0)$

Coordinates of $N = \left(\frac{-2 + 7}{2}, \frac{3 + 2}{2}\right)$
$= \left(\frac{5}{2}, \frac{5}{2}\right)$

(b) Area of triangle $CMN = \frac{1}{2}\begin{vmatrix} -2 & -3 & \frac{5}{2} & -2 \\ 3 & 0 & \frac{5}{2} & 3 \end{vmatrix}$

$= \frac{1}{2}\left|\left(0 - \frac{15}{2} + \frac{15}{2}\right) - \left(-9 + 0 - 5\right)\right|$

$= \frac{1}{2}|14|$

$= 7 \text{ units}^2$

Area of quadrilateral $ABNM = \frac{1}{2}\begin{vmatrix} -4 & 7 & \frac{5}{2} & -3 & -4 \\ -3 & 2 & \frac{5}{2} & 0 & -3 \end{vmatrix}$

$= \frac{1}{2}\left|\left(-8 + \frac{35}{2} + 0 + 9\right) - \left(-21 + 5 - \frac{15}{2} + 0\right)\right|$

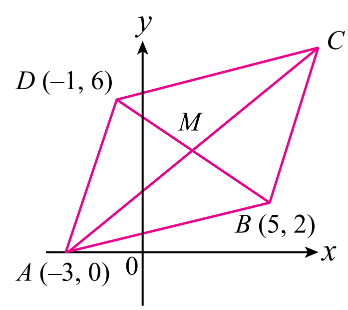
$= \frac{1}{2}|42|$

$= 21 \text{ units}^2$

Thus, the ratio of area of triangle *CMN* to quadrilateral *ABNM* is 7 : 21 = 1 : 3.

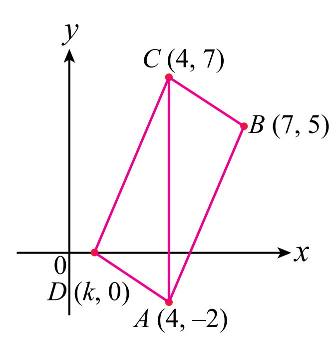
Self Practice 7.9

1. In the diagram on the right, points $A(-3, 0)$, $B(5, 2)$, C and $D(-1, 6)$ are the vertices of a parallelogram *ABCD*. *M* is the intersection point of diagonals *AC* and *BD*. Find
(a) the coordinates of *C* and *M*,
(b) the ratio of the area of ΔABM to the area of parallelogram *ABCD*.



2. The straight line $y = 8 - 2x$ intersects the straight line $y = k$, x-axis and y-axis at points *P*, *Q* and *R* respectively. Given the area of ΔOPR is 12 units², with *O* as the origin, find
(a) the smallest value of k,
(b) the coordinates of *P*.

3. In the diagram on the right, *ABCD* is a parallelogram with vertices $A(4, -2)$, $B(7, 5)$, $C(4, 7)$ and $D(k, 0)$. Find
(a) the area of ΔABC,
(b) the value of k if the area of ΔACD is equal to the area of ΔABC,
(c) the coordinates of *E* if *ACBE* is a parallelogram,
(d) the area of parallelogram *ACBE*.

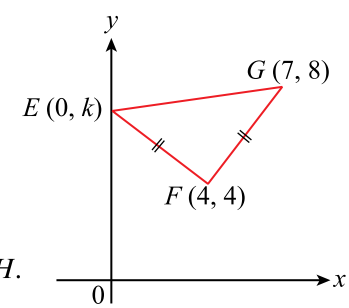


CHAPTER 7

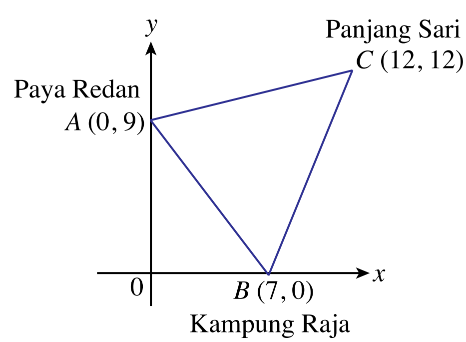
Intensive Practice 7.3

Scan the QR code or visit bit.ly/33hpduP for the quiz

1. *ABCD* is a parallelogram which diagonals intersect at *E*. Given $A(-5, 3)$, $B(0, -2)$ and $C(3, 5)$, find
   (a) the coordinates of *D* and *E*,
   (b) the area of parallelogram *ABCD*.
2. *PQRS* is a rhombus with coordinates $P(3, 3)$, $Q(h, 3)$, $R(-5, -1)$ and $S(0, k)$. Find
   (a) the value of *h* and *k*,
   (b) the area of rhombus *PQRS*.
3. Given three points, $A(-1, -5)$, $B(2, 1)$ and $C(6, 9)$,
   (a) find the area of ΔABC,
   (b) based on the answer in (a), what can you say about points *A, B* and *C*?
4. Find the area of a polygon with vertices $(5, 2)$, $(-1, -3)$, $(2, 6)$, $(3, -2)$, $(-4, 0)$ and $(-3, 2)$.
5. Points $A(5, -1)$, $B(3, 3)$ and $C(-6, p)$ are the vertices of a triangle. Find the values of *p* if the area of ΔABC is 16 units^2.
6. Given three points, $P(2, 2r - 1)$, $Q(r - 1, r + 1)$ and $R(r + 3, 0)$. If points *P*, *Q* and *R* are on the same straight line, find the possible values of *r*.
7. Three points have coordinates $A(8, a)$, $B(-1, 2)$ and $C(3, 10)$. Find the value of *a* if
   (a) *A*, *B* and *C* are collinear,
   (b) the area of ΔABC is 12 units^2.
8. The diagram on the right shows an isosceles triangle *EFG* with vertices $E(0, k)$, $F(4, 4)$ and $G(7, 8)$. *EF* and *FG* have the same length.
   (a) Find the value of *k*.
   (b) *H* is a point on the line $y = 11$ such that $EH = GH$. Find
      (i) the coordinates of *H*,
      (ii) the ratio of area of ΔEFG to area of quadrilateral *EFGH*.

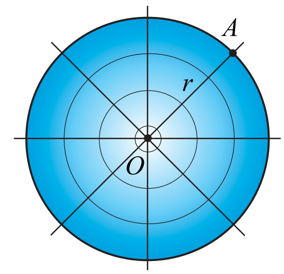


9. Points $O(0, 0)$, $P(m + 1, m - 7)$, $Q(2m + 1, 2m)$ and $R(m, m + 6)$ are the vertices of a quadrilateral such that $m > 0$.
   (a) If the area of *OPQR* is $34\frac{1}{2}$ units^2, find the value of *m*.
   (b) Hence, find the area of ΔOPR.
10. The coordinates of three LRT stations, Paya Redan, Kampung Raja and Panjang Sari are represented by points $A(0, 9)$, $B(7, 0)$ and $C(12, 12)$, such that 1 unit represents 100 m. Find
    (a) the distance, in km, between Paya Redan Station and Kampung Raja Station.
    (b) the actual area, in km^2, of the triangle formed by the three stations.



7.4 Equations of Loci

The locus of a moving point is the path taken by the point subject to certain conditions. For example, the path traced by point A moving r units from the fixed point O on the radar screen at an air traffic control centre as shown in the diagram on the right is a locus in the shape of a circle and it can be represented by an equation. Can you determine the equation of the locus of moving point A which is a circle?



Determining the equation of locus

The equation of the locus of a moving point which satisfies certain conditions can be determined using the formula for distance between two points or other formulas depending on the given conditions.

A Locus of a moving point from a fixed point is constant

INQUIRY 4 In groups

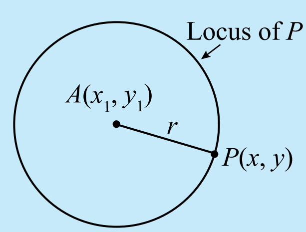
Aim: To explore the shape and determine the equation of locus of a moving point which the distance from a fixed point is constant

Instructions:
1. Scan the *QR* code or visit the link on the right.
2. Let $P(x, y)$ is a point of distance r from a fixed point $A(x_1, y_1)$ where $r > 0$.
3. Move point P and observe the path traced by point P.
4. What is the shape of the locus of point P obtained?
5. Using the formula of distance between two points, write the equation for the shape obtained in terms of x, y, x_1, y_1 and r.

bit.ly/2pMIhTe

From the results of Inquiry 4, the shape of locus of point P obtained is a circle with centre $A(x_1, y_1)$ and radius r unit. Equation of the locus of moving point $P(x, y)$ which the distance from a fixed point $A(x_{1,}\ y_1)$ is always constant, can be determined using the distance formula as follows:

$$PA = r$$
$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = r$$
$$(x - x_1)^2 + (y - y_1)^2 = r^2, \text{ where } r > 0$$



CHAPTER 7

Example 13

Find the equation of locus of moving point P so that its distance from point $A(4, –3)$ is 6 units.

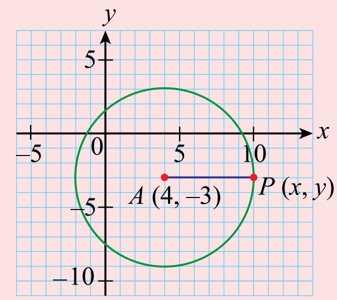
Solution

Let the coordinates of point P are (x, y).
Distance of P from $A = 6$

$$\sqrt{(x-4)^2 + [y-(-3)]^2} = 6$$
$$(x-4)^2 + (y+3)^2 = 36$$
$$x^2 - 8x + 16 + y^2 + 6y + 9 = 36$$
$$x^2 + y^2 - 8x + 6y - 11 = 0$$

Square both sides of equation

Thus, equation of locus of P is $x^2 + y^2 - 8x + 6y - 11 = 0$.



B Ratio of distance of moving point from two fixed points is constant

INQUIRY 5 In groups

Aim: To explore the shape and determine the equation of locus of a moving point whose the ratio of the distance from two fixed points is constant

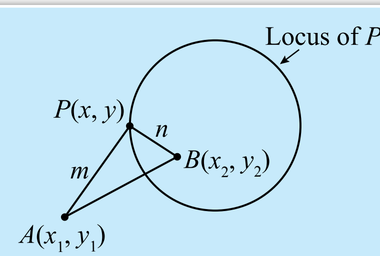
bit.ly/2pMljuk

Instructions:

1. Scan the QR code or visit the link on the right.
2. Let $P(x, y)$ is a moving point such that its distance from two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ is in the ratio $m : n$, that is $\frac{PA}{PB} = \frac{m}{n}$.
3. Drag the point on the slider to the left and right so that ratio r changes and observe the circle formed.
4. Does the locus of moving point P form a circle? If yes, can you determine its equation in terms of $x, y, x_1, y_1, x_2, y_2, m$ and n?
5. Then, drag the slider r to the left again so that its value is 1, that is, $PA : PB = 1 : 1$.
6. Make a conjecture about the shape of locus of moving point P that will be obtained if $PA = PB$. Can you determine its equation?

From the results of Inquiry 5, the shape of locus of moving point P is a circle and the equation of locus of moving point $P(x, y)$, which the distance from two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$, is always constant in the ratio $m : n$ can be determined by using the distance formula as follows:

$$\frac{PA}{PB} = \frac{m}{n}$$
$$\frac{\sqrt{(x-x_1)^2 + (y-y_1)^2}}{\sqrt{(x-x_2)^2 + (y-y_2)^2}} = \frac{m}{n}$$
$$\frac{(x-x_1)^2 + (y-y_1)^2}{(x-x_2)^2 + (y-y_2)^2} = \frac{m^2}{n^2}$$



7.4.1

CHAPTER 7

When $PA : PB = 1 : 1$, $P(x, y)$ is always the same distance from two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$, the shape of locus of P is the perpendicular bisector of line AB. Its equation is:

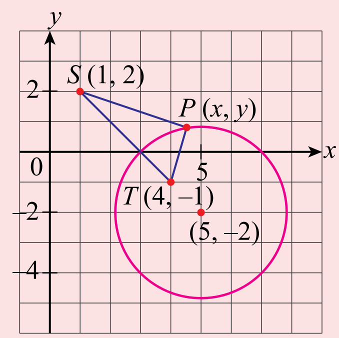
$$PA = PB$$
$$\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}$$
$$(x-x_1)^2+(y-y_1)^2 = (x-x_2)^2+(y-y_2)^2$$

Example 14

Point P moves such that its distance from points $S(1, 2)$ and $T(4, -1)$ is in the ratio $2 : 1$. Find the equation of locus of moving point P.

Solution

Let $P(x, y)$ is a moving point.



$$\frac{PS}{PT} = \frac{2}{1}$$

$$\frac{\sqrt{(x-1)^2+(y-2)^2}}{\sqrt{(x-4)^2+(y+1)^2}} = \frac{2}{1}$$ ← Square both sides of equation

$$\frac{(x-1)^2+(y-2)^2}{(x-4)^2+(y+1)^2} = \frac{4}{1}$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 = 4(x^2 - 8x + 16 + y^2 + 2y + 1)$$
$$x^2 + y^2 - 2x - 4y + 5 = 4x^2 + 4y^2 - 32x + 8y + 68$$
$$3x^2 + 3y^2 - 30x + 12y + 63 = 0$$ ← Divide each term by 3
$$x^2 + y^2 - 10x + 4y + 21 = 0$$

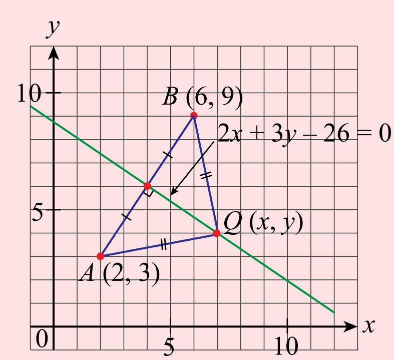
Thus, the equation of locus of moving point P is $x^2 + y^2 - 10x + 4y + 21 = 0$.

Example 15

Find the equation of locus of a moving point Q such that its distance from point $A(2, 3)$ and point $B(6, 9)$ are the same.

Solution

Let $Q(x, y)$ is a moving point.



$$QA = QB$$
$$\sqrt{(x-2)^2+(y-3)^2} = \sqrt{(x-6)^2+(y-9)^2}$$
$$(x-2)^2+(y-3)^2 = (x-6)^2+(y-9)^2$$
$$x^2 - 4x + 4 + y^2 - 6y + 9 = x^2 - 12x + 36 + y^2 - 18y + 81$$
$$8x + 12y - 104 = 0$$
$$2x + 3y - 26 = 0$$

Thus, the equation of locus of moving point Q is $2x + 3y - 26 = 0$.

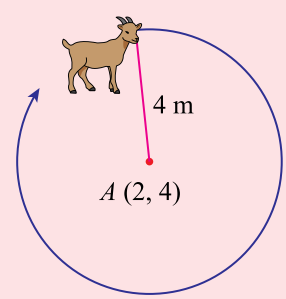
Self Practice 7.10

1. Find the equation of locus of moving point P such that its distance from each of the following points is 3 units.
   (a) $(0, 0)$ (b) $(2, 3)$ (c) $(-4, 5)$ (d) $(-1, -6)$
2. Point P moves such that its distance from $Q(-2, 1)$ is always 5 units. Find the equation of locus of point P.
3. Find the equation of locus of moving point P such that its distance from the following fixed points is in the ratio given.
   (a) $A(-2, 0)$, $B(4, 0)$; ratio $1 : 2$ (b) $C(-3, 0)$, $D(2, 5)$; ratio $1 : 3$
   (c) $E(0, 2)$, $F(-2, 4)$; ratio $3 : 2$ (d) $R(1, 2)$, $S(4, -1)$; ratio $2 : 1$
4. The coordinates of points J and K are $(-1, 3)$ and $(4, 6)$ respectively. Point Q moves such that $QJ : QK = 2 : 3$. Find the equation of locus of Q.
5. Point R moves such that its distance from point $A(6, 0)$ is twice its distance from point $B(-3, 0)$. Find the equation of locus of R.
6. Point P moves in the ratio $PO : PA = 1 : 4$, with O the origin and coordinates of point A being $(2, 0)$. Find the equation of locus of point P.
7. Find the equation of locus of moving point P such that its distances from the following points are the same.
   (a) $A(-2, 0)$ and $B(0, 4)$ (b) $C(-3, 5)$ and $D(2, -4)$ (c) $J(2, 3)$ and $K(6, 8)$

Solving problems involving equations of loci

Example 16 MATHEMATICS APPLICATION

A goat is tied with a rope to a pole which is planted in the middle of a field. The length of rope used is 4 metres. The goat walks around the pole tied to the end of the taut rope as shown in the diagram. If the coordinates of the pole are $A(2, 4)$, find the equation of locus of the goat's track.



Solution

1. Understanding the problem

- A goat is tied with a rope 4 metres long to the pole.
- Coordinates of the pole are $A(2, 4)$.
- Find the equation of locus of goat's track around the pole with the rope taut.

2. Planning the strategy

- Track of the goat is a circle with centre $A(2, 4)$ and radius 4 metres.
- Use formula for distance between two points, $d = \sqrt{(x - x_1)^2 + (y - y_1)^2}$ to find the equation of locus of goat's track.

7.4.1 7.4.2

3. Implementing the strategy

Let $P(x, y)$ is a moving point at the end of rope tied to the goat's neck.

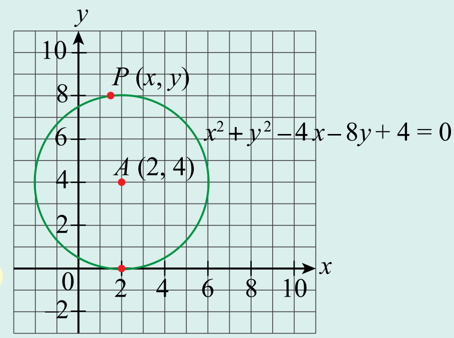
$$PA = 4$$
$$\sqrt{(x-2)^2 + (y-4)^2} = 4$$
$$(x-2)^2 + (y-4)^2 = 16$$
$$x^2 - 4x + 4 + y^2 - 8y + 16 = 16$$
$$x^2 + y^2 - 4x - 8y + 4 = 0$$

Thus, the equation of locus of moving point P, which is the goat's track around the pole with the rope taut is $x^2 + y^2 - 4x - 8y + 4 = 0$.

4. Making a conclusion

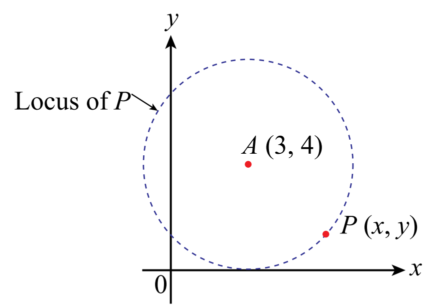
Represent the equation of locus of P on the Cartesian plane. It is found that the locus of P touches the x-axis at point $(2, 0)$. Substitute $(2, 0)$ into equation of locus of P.

Left side $= 2^2 + 0^2 - 4(2) - 8(0) + 4$
$= 0$
$=$ right side



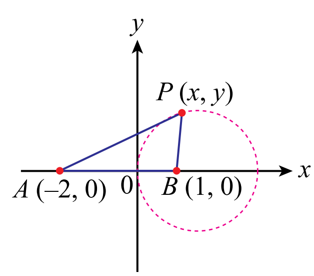
Self Practice 7.11

1. The diagram on the right shows the locus of a moving point $P(x, y)$ that touches the x-axis at a point and is of fixed distance from a point $A(3, 4)$. Find the equation of locus of P.



2. Point P moves such that it is always the same distance from the points $Q(8, 7)$ and $R(11, 4)$. Point S moves such that its distance from point $T(7, 8)$ is always 5 units. The locus of point P and the locus of point S intersect at two points.
   (a) Find the equation of locus of point P.
   (b) Show that the locus of point S is $x^2 + y^2 - 14x - 16y + 88 = 0$.
   (c) Find the coordinates of the intersection points of the two loci.

3. In the diagram on the right, point $A(-2, 0)$ and point $B(1, 0)$ are two fixed points. Point P moves along the circle such that the ratio $PA : PB = 2 : 1$. Show that
   (a) the equation of the circle is $x^2 + y^2 - 4x = 0$,
   (b) point $C(2, 2)$ is on the circle.

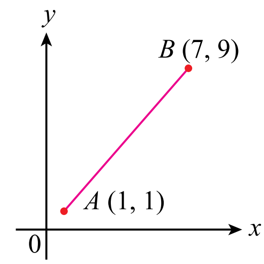


Intensive Practice 7.4

Scan the QR code or visit bit.ly/33hpOg3 for the quiz

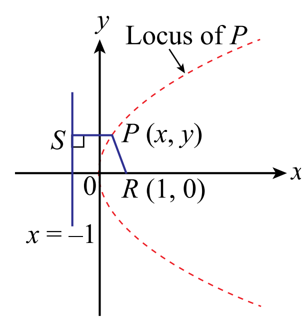
1. A point $R(x, y)$ moves so that its distance from two fixed points $A(-1, 10)$ and $B(2, 6)$ is such that $\frac{RA}{RB} = \frac{1}{2}$. Find
(a) the equation of locus of R,
(b) the coordinates of the point on locus R that touches the y-axis.

2. The diagram on the right shows a line segment AB with coordinates $A(1, 1)$ and $B(7, 9)$. Find the equation of locus of moving point S such that triangle ABS always has a right angle at S.

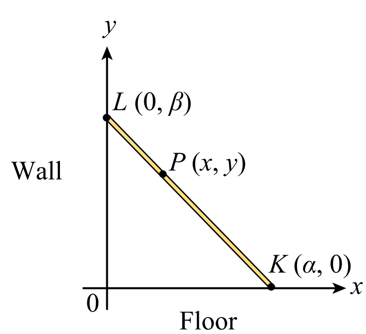


3. Point Q moves along the arc of a circle with centre $(6, 5)$. The arc of the circle passes through $R(2, 8)$ and $S(k, 2)$. Find
(a) the equation of locus of Q,
(b) the values of k.

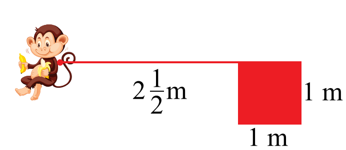
4. The diagram on the right shows the locus of moving point P such that its distance from point $R(1, 0)$ and line $x = -1$ are the same. Find the equation of locus of moving point P.



5. The diagram on the right shows the x-axis and y-axis which represent the floor and wall respectively. A pole, LK of length 9 m leaning against the wall touches the floor and wall at points $K(\alpha, 0)$ and $L(0, \beta)$ respectively.
(a) Write the equation which relates α and β.
(b) Given $P(x, y)$ is a point on the pole such that the ratio $LP : PK = 1 : 2$. Both ends of the pole slide along the x-axis and y-axis. Find the equation of locus of point P.



6. A monkey is tied to one vertex of his cage which measures 1 m × 1 m with a rope. The length of the rope is $2\frac{1}{2}$ m. Sketch and explain the locus if the monkey moves anticlockwise around the cage with the rope taut.



CHAPTER 7

SUMMARY OF CHAPTER 7

COORDINATE GEOMETRY

Divisor of line segment

$P(x, y) = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}\right)$

Parallel lines and perpendicular lines

Lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if and only if $m_1 = m_2$

Lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if and only if $m_1m_2 = -1$

Equation of locus

- Locus of moving point $P(x, y)$ which distance is constant from fixed point $A(x_1, y_1)$ is $(x - x_1)^2 + (y - y_1)^2 = r^2$
- Locus of moving point $P(x, y)$ which distance from two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ is always constant in the ratio $m : n$ is
$$\frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_2)^2 + (y - y_2)^2} = \frac{m^2}{n^2}$$
- Locus of moving point $P(x, y)$ which distance from two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the same is $(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$

Area of polygon

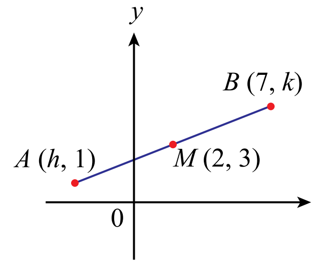
$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n & x_1 \\ y_1 & y_2 & y_3 & \cdots & y_n & y_1 \end{vmatrix}$$

$$= \frac{1}{2}|x_1y_2 + x_2y_3 +, \ldots, + x_ny_1 - x_2y_1 - x_3y_2 -, \ldots, - x_1y_n|$$

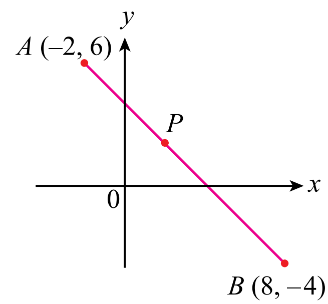
WRITE YOUR JOURNAL

Coordinate geometry has introduced the general form, gradient form, intercept form and other forms in expressing the equation of a straight line. What are the advantages of expressing the equation in these forms? Which form do you prefer to use? Why?

MASTERY PRACTICE



1. The diagram on the right shows a straight line AB. The midpoint of the line joining $A(h, 1)$ and $B(7, k)$ is $M(2, 3)$. Find PL3
(a) the value of h and k,
(b) the gradient of the line,
(c) the equation of the perpendicular bisector of AB.

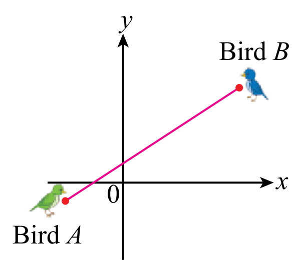


2. Given a straight line AB with points $A(-2, 6)$ and $B(8, -4)$. Point P lies on AB such that $AP : PB = 2 : 3$. Find PL3
(a) the coordinates of point P,
(b) the equation of the straight line that is perpendicular to AB and passes through point P.

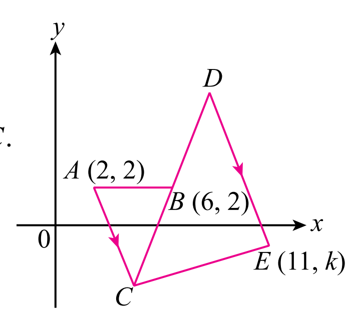
3. Given three points $P(1, -1)$, $Q(n, 2)$ and $R(n^2, n + 3)$. If points P, Q and R lie on the same straight line, find the possible values of n. PL3

4. Given two points $R(-3, 4)$ and $S(3, -1)$. Find the coordinates of point T that might be on the y-axis such that area of ΔRST is 13.5 unit^2. PL3

5. Point $P(x, y)$ moves such that its distance from point $A(2, 0)$ is three times its distance from point $B(-4, 0)$. Find the equation of locus of point P. PL3

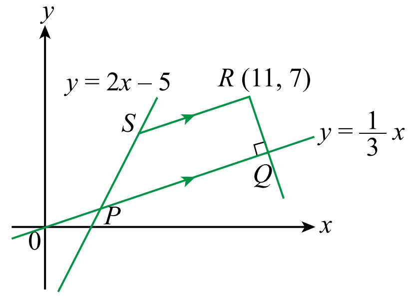


6. The diagram on the right shows the position of two birds, A and B on the Cartesian plane. The coordinates of birds A and B are $(-3, -1)$ and $(6, 5)$ respectively. The two birds fly towards each other in a straight line with different velocities. The velocity of bird A is twice the velocity of bird B. Find the coordinates where the two birds meet. PL3



7. The diagram on the right shows an isosceles triangle ABC such that the coordinates of A are $(2, 2)$, coordinates of B are $(6, 2)$ and C is below the x-axis. PL3
(a) Given the area of ΔABC is 10 unit^2, find the coordinates of C.
(b) Line CB is extended to point D so that B is the midpoint of CD. Find the coordinates of D.
(c) A line is drawn from point D, parallel to AC, to point $E(11, k)$ and C is joined to E.
 (i) Find the value of k.
 (ii) Show that CED is not a right-angled triangle.

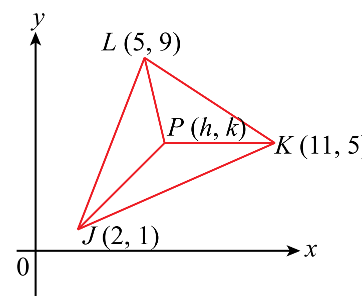
8. In the diagram on the right, $PQRS$ is a trapezium with PQ parallel to SR and $\angle PQR = 90°$. The coordinates of vertex R are $(11, 7)$. The equation of PQ and PS are $y = \frac{1}{3}x$ and $y = 2x - 5$ respectively. Find PL4



(a) the coordinates of P,
(b) the equation of QR and SR,
(c) the coordinates of Q and S,
(d) the area of trapezium $PQRS$.

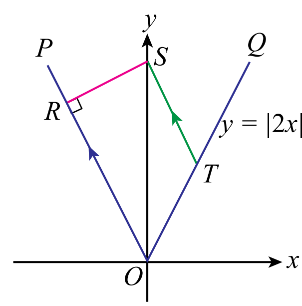
Hence, show that $\frac{\text{area } \Delta PQR}{\text{area } \Delta PRS} = \frac{PQ}{SR}$.

9. The coordinates of the vertices of ΔJKL are $J(2, 1)$, $K(11, 5)$ and $L(5, 9)$. Point $P(h, k)$ is in the triangle such that all the areas of ΔJKP, ΔKLP and ΔJLP are the same. PL5



(a) Find the area of ΔJKL.
(b) Express the area of ΔJKP and ΔKLP in terms of h and k.
(c) Find the coordinates of P.
(d) Find the equation of the line JP.
(e) If JP is extended to meet KL at Q, find
  (i) the coordinates of Q,
  (ii) the ratio of $KQ : QL$.

10. In the diagram on the right, POQ is the graph of $y = |2x|$. R is a point on OP such that $OR = \sqrt{45}$ unit and O is the origin. RS is perpendicular to OP and OR is parallel to TS. Find PL5

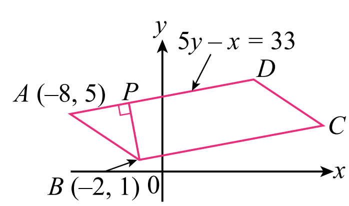


(a) the coordinates of R, S and T,
(b) the area of trapezium $ORST$.

11. $P(h, 8)$ and $Q(k, 2)$ are two points on the curve $y = \frac{8}{x}$. PL5

(a) Find the value of h and k.
(b) Find the equation of PQ.
(c) Using coordinate geometry methods, find the equations of tangent to the curve that are parallel to PQ.

12. The diagram on the right shows a parallelogram $ABCD$ with the coordinates of A and B being $(-8, 5)$ and $(-2, 1)$ respectively. The equation of AD is $5y - x = 33$. Line PB is perpendicular to line AD and $AP : PD = 1 : 2$. Find PL4

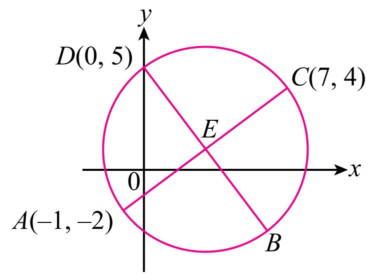


(a) the equation of BP,
(b) the coordinates of P, D and C,
(c) the area of parallelogram $ABCD$.

13. In the diagram on the right, AC and BD are the diameter of the circle with centre E. Points A, C and D are $(-1, -2)$, $(7, 4)$ and $(0, 5)$ respectively. PL5

(a) Find the coordinates of E and B.

(b) What type of quadrilateral is $ABCD$?



14. Early of each month, a magazine publishing company sells x copies of magazines at RM6.00 per copy. The cost for one copy of magazine is RM2.00 and early every month, the company pays a fixed cost of RM400 for printing, storage and delivery. PL5

(a) Write the equation which relates profit, P, in RM, to the number of copies, x of magazines sold.

(b) Draw a graph for the equation obtained. From the graph drawn,
  (i) find the profit gained if 500 copies of magazines are sold,
  (ii) calculate the number of copies of magazines sold if the profit gained is RM1 000.

15. The vertices of a triangle ABC are $A(1, 2)$, $B(6, 7)$ and $C(7, 2)$. Draw triangle ABC and construct the perpendicular bisector of AB, BC and CA. Mark the intersection point as P. What can you say about the intersection point? Draw a circle with centre P and radius AP. What can you say about the circle? Repeat the same procedures for other triangles in order to verify your answer. PL6

Exploring MATHEMATICS

1. The equation $y = mx$ where m is the gradient, defines a group of lines, that is one line for every value of m.

(a) Using dynamic geometry software, draw the graphs for the group of lines when its gradient is zero, $m = 0$, followed by positive gradient, that is $m = \frac{1}{2}$, $m = 1$, $m = 2$ and $m = 6$, then negative gradient, that is $m = -\frac{1}{2}$, $m = -1$, $m = -2$ and $m = -6$.

(b) From the graphs obtained, what happens to the magnitude of gradient of the lines when the graphs get nearer to the vertical line? Can you draw a conclusion about each line in the group of lines?

2. The equation $y = 2x + c$ defines a group of lines, that is a line for every value of c.

(a) Using dynamic geometry software, draw the graph for the group of lines when $c = -6$, $c = -3$, $c = 0$ and $c = 6$.

(b) From the graph drawn, what can you conclude about every line of the group of lines?

CHAPTER 8 Vectors

What will be learnt?

- Vectors
- Addition and Subtraction of Vectors
- Vectors in Cartesian Plane

List of Learning Standards

bit.ly/33rCukN

The flight system in Malaysia connects people to different destinations around the world. The company extended flight routes to more than 1 000 destinations, involving around 150 countries. In your opinion, what kind of information is needed for a pilot to make sure that the suitable route is selected for the intended destination?

KEYWORDS

Vector	*Vektor*
Magnitude	*Magnitud*
Direction	*Arah*
Directed line segment	*Tembereng garis berarah*
Zero vector	*Vektor sifar*
Negative vector	*Vektor negatif*
Collinear	*Segaris*
Resultant vector	*Vektor paduan*
Position vector	*Vektor kedudukan*
Triangle law	*Hukum segi tiga*
Parallelogram law	*Hukum segi empat selari*
Polygon law	*Hukum poligon*

Did you Know?

For a quantity that involves magnitude and direction, vector is applied widely in the fields of mathematics and physics. Other than that, vector is also applied in daily life such as in navigations, computer science, geometry and topology fields.

For further information:

bit.ly/2MCJKUa

SIGNIFICANCE OF THIS CHAPTER

Knowledge of vector is important because of its application in the field of mathematics and physics. In the branch of mechanics, vector is used to represent quantity such as displacement, force, weight, velocity and momentum. Vector is also widely used in sailing and flight.

Scan this QR code to watch a video about Malaysia Airlines.

bit.ly/2LazgOc

8.1 Vectors

Comparing the differences and identifying vector and scalar

In our daily lives, there are various quantities with magnitude and direction and quantities with magnitude but without direction. A quantity that has magnitude and direction is called a **vector quantity**, while a quantity with magnitude but no direction is called a **scalar quantity**.

Observe the following situations carefully.

Temperature of a liquid inside a refrigerator is –12°C.

A moving car on the road is heading south with speed of 80 km h^{-1}.

Can you determine which situation involves a vector quantity and a scalar quantity? How do you identify whether the quantity is a vector quantity or a scalar quantity?

The following table shows the examples of quantities with vector and scalar and quantities without both vector and scalar.

Vector	Scalar	Without vector and scalar
50 N of force applied to a box.	Auni's height is 1.48 m.	Pressure and tension
Velocity of a car is 90 km h^{-1} and is heading east.	The area of a tile is 120 cm^2.	Conductivity of metal

QR

Additional information about vector and scalar.

bit.ly/2OCQiVm

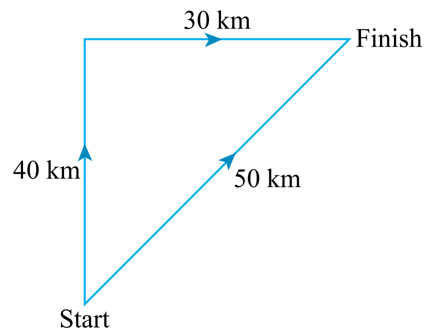
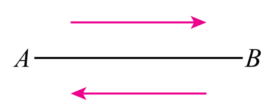
Mind Challenge

List a few situations that involve vector and scalar and list a few situations that do not involve vector or scalar.

BRAINSTORMING

Scalar quantity is tensor at level zero while scalar vector is tensor at level one. Surf the Internet to look for further information regarding to tensor and discuss your findings.

Can you differentiate between distance and displacement, speed and velocity and also mass and weight? The following table shows the differences between those quantities.

Scalar quantity	Vector quantity	Examples
Distance Total length of track that is passed by a moving object.	**Displacement** The length of the shortest line segment between the starting point and the terminal point and involves the direction from a reference point.	 A car moves 40 km to the North and 30 km to the East. Distance = 40 km + 30 km = 70 km Displacement = 50 km
Speed Rate of change of distance against time.	**Velocity** Rate of change of displacement against time. The value becomes negative if the object is moving in the opposite direction.	 Haziq is moving from *A* to *B* with the same speed and velocity, which is 90 km h^{-1}. After that he turns back from *B* to *A* with speed of 90 km h^{-1} and velocity of –90 km h^{-1}.
Mass Amount of matter that an object contains. The value does not change according to the location.	**Weight** The pull of gravity on an object. The value is not constant and depends on the gravitational force of the location.	The mass of an astronaut when he is on the moon is 120 kg with weight of 200 N while the mass of the astronaut is 120 kg and weight is 1200 N when he is on earth.

Example 1

State whether each of the following quantities is a vector quantity or a scalar quantity. Justify your answer.

(a) Mikail walks 1 km from home to the grocery store.
(b) A car is moving at a speed of 90 km h^{-1} to the south.
(c) The body temperature of Alicia hits 38°C.

Solution

(a) Scalar quantity because the quantity only consists of magnitude.
(b) Vector quantity because the quantity consists of magnitude and direction.
(c) Scalar quantity because the quantity only consists of magnitude.

Self Practice 8.1

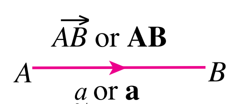
1. State whether each of the following quantities is a scalar quantity or a vector quantity. Justify your answer.

(a) Liquid *X* with a density of 1.2 g cm^{-3}.
(b) A box weight 150 N is moved as high as 1 m from the floor.
(c) The volume of a mineral water bottle is 1.5 *l*.
(d) The duration of holiday for Suzie is 3 days and 2 nights.
(e) A ball is given a horizontal impulse of 5.0 Ns.

Representing the vector using directed line segment and vector notations and determining the magnitude and direction of vectors

Vector can be represented using a line segment with an arrow or better known as directional line segment. The arrow represents the direction of the vector while the length of the line represents the magnitude of the vector.

As an example, vector of a sailing boat moves 7 km to the east from point *A* to point *B* can be represented by the directional line segment as shown in the diagram on the right. Point *A* is the starting point and Point *B* is the terminal point.



Vector can also be represented with the notation as below:

$$\overrightarrow{AB} \text{ or } \mathbf{AB} \text{ or } \underset{\sim}{a} \text{ or } \mathbf{a}$$

Magnitude of the vector can be written as:

$$|\overrightarrow{AB}| \text{ or } |\mathbf{AB}| \text{ or } |\underset{\sim}{a}| \text{ or } |\mathbf{a}|$$

Zero vectors are vectors that consist of zero magnitude and the direction cannot be determined. Zero vectors can be represented as $\underset{\sim}{0}$.

Example:

A race car is moving in the round-shaped track. The starting point and the terminal point of the moving car are the same. Thus, the vector for displacement of the race car is zero vectors.

Two vectors are the same if both vectors consist of same magnitude and direction, $\overrightarrow{AB} = \overrightarrow{CD}$.

Example:

Rakesh and Fauzi are riding bicycles at the same speed and in the same direction. The velocity vector, v, for both movements are the same. Thus, $v_{\text{Rakesh}} = v_{\text{Fauzi}}$.

A vector is negative if the vector consists of the same magnitude but is moving in the opposite direction. Vector $\overrightarrow{BA}$ is a negative vector of vector $\overrightarrow{AB}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Example:

Two trains, A and B move pass each other at two parallel railways at the same speed but different direction. The velocity of train A has a positive value while the velocity of train B has a negative value.

Example 2

Draw and label the following vectors.

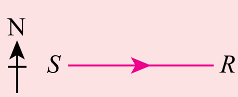
(a) $\overrightarrow{SR}$ represents displacement of 12 km to the east.

(b) $\underset{\sim}{p}$ represents force of 7 N to the south.

(c) $\mathbf{r}$ represents velocity of 70 m s^{-1} to the left.

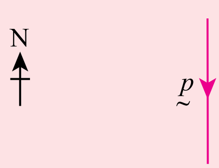
Solution

(a)



1 cm represents 6 km

(b)



1 cm represents 3.5 N

(c)

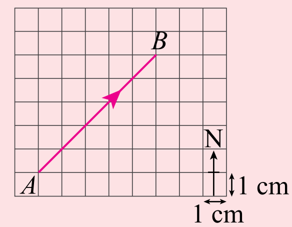
1 cm represents 20 m s^{-1}

CHAPTER 8

8.1.2

Example 3

The diagram on the right shows vector $\overrightarrow{AB}$ that represents the displacement of a particle from point A to point B. Find the magnitude and direction of the particle from point A.



Solution

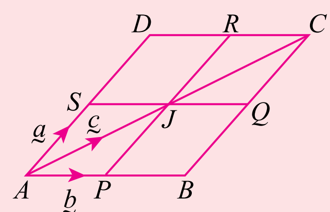
$$|\overrightarrow{AB}| = \sqrt{5^2 + 5^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

Thus, the magnitude of $\overrightarrow{AB}$ is $5\sqrt{2}$ cm and the direction of $\overrightarrow{AB}$ is to the Northeast.

Example 4

The diagram on the right shows a parallelogram, $ABCD$. The points P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. Given that $\overrightarrow{AS} = \underset{\sim}{a}$, $\overrightarrow{AP} = \underset{\sim}{b}$ and $\overrightarrow{AJ} = \underset{\sim}{c}$. State the vectors for the following in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ or $\underset{\sim}{c}$.

(a) $\overrightarrow{SD}$ (b) $\overrightarrow{CJ}$ (c) $\overrightarrow{RJ}$ (d) $\overrightarrow{JQ}$



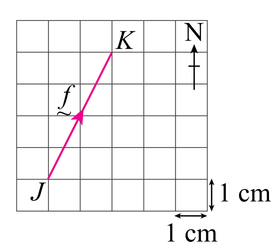
Solution

(a) $\overrightarrow{SD} = \underset{\sim}{a}$ (b) $\overrightarrow{CJ} = -\underset{\sim}{c}$ (c) $\overrightarrow{RJ} = -\underset{\sim}{a}$ (d) $\overrightarrow{JQ} = \underset{\sim}{b}$

Self Practice 8.2

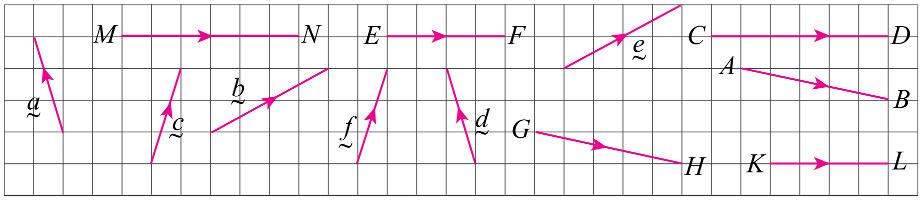
1. By using suitable scales, draw and label the following vectors.
   (a) $\overrightarrow{XY}$ represents 5 N of force to the right.
   (b) **RS** represents 40 km of displacement to the southwest.
   (c) $\underset{\sim}{v}$ represents velocity of 20 km h^{-1} to the west.
   (d) **a** represents momentum of 7 kg m s^{-1} to the left.

2. The diagram on the right shows vector $\underset{\sim}{f}$ that represents the force applied on an object from J to K. Find the magnitude and the direction of vector $\underset{\sim}{f}$.

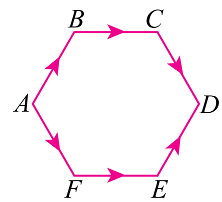


3. Two cars, A and B are moving away from town O. Car A moves to the north while car B moves to the east. Find the distance between the two cars after both cars travelled for one hour, given that $|\overrightarrow{OA}| = 90$ km and $|\overrightarrow{OB}| = 75$ km.

4. Find pairs of the same vectors in the diagram below.



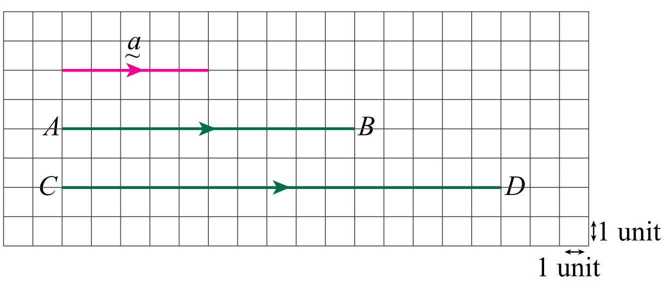
5. The diagram on the right shows a regular hexagon *ABCDEF*.
(a) State the same vector for
(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{BC}$ (iii) $\overrightarrow{CD}$
(b) State the negative vector for
(i) $\overrightarrow{AF}$ (ii) $\overrightarrow{FE}$ (iii) $\overrightarrow{ED}$



Making and verifying conjectures about the properties of scalar multiplication on vectors

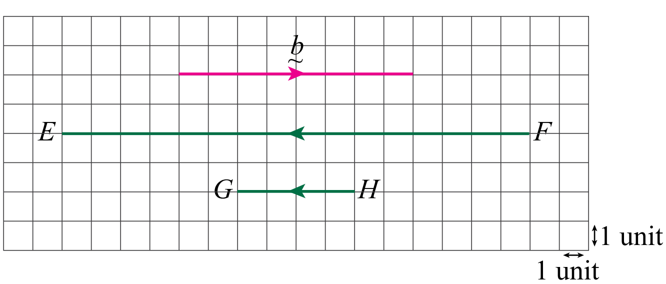
Observe the two following cases:

Case 1



Observe vector $\underset{\sim}{a}$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in the diagram on the left.
It is found that $\overrightarrow{AB} = 2 \times \underset{\sim}{a}$ or $2\underset{\sim}{a}$, and
$\overrightarrow{CD} = 3 \times \underset{\sim}{a}$ or $3\underset{\sim}{a}$
Given that $|\underset{\sim}{a}| = 5$ units, thus
$|\overrightarrow{AB}| = 10$ units and $|\overrightarrow{CD}| = 15$ units.

Case 2



Observe vector $\underset{\sim}{b}$, $\overrightarrow{EF}$ and $\overrightarrow{GH}$ in the diagram on the left.
It is found that $\overrightarrow{EF} = 2 \times (-\underset{\sim}{b})$ or $(-2)\underset{\sim}{b}$,
and $\overrightarrow{GH} = \frac{1}{2} \times (-\underset{\sim}{b})$ or $\left(-\frac{1}{2}\right)\underset{\sim}{b}$
Given that $|\underset{\sim}{b}| = 8$ units, thus
$|\overrightarrow{EF}| = 16$ units and $|\overrightarrow{GH}| = 4$ units.

CHAPTER 8

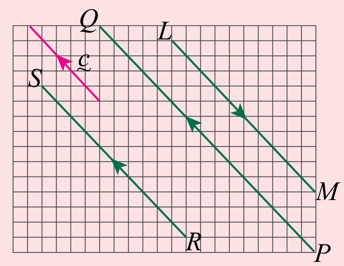
8.1.2 8.1.3

From case 1 and case 2, we can conclude that:

Multiplication of scalar k with vector $\underset{\sim}{a}$ produces vector $k\underset{\sim}{a}$, with the conditions:
(a) $|k\underset{\sim}{a}| = k|\underset{\sim}{a}|$.
(b) $k\underset{\sim}{a}$ is in the same direction with $\underset{\sim}{a}$ if $k > 0$.
(c) $k\underset{\sim}{a}$ is in the opposite direction with $\underset{\sim}{a}$ if $k < 0$.

Example 5

State the following vectors in the diagram on the right in terms of $\underset{\sim}{c}$.

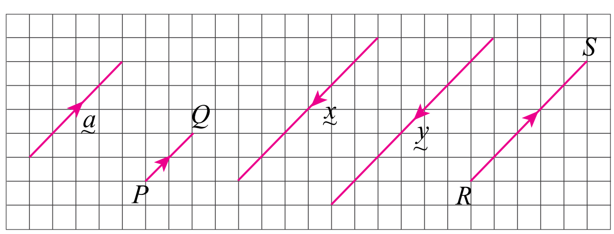


Solution

$\overrightarrow{RS} = 2\underset{\sim}{c}$, $\overrightarrow{PQ} = 3\underset{\sim}{c}$, $\overrightarrow{LM} = -2\underset{\sim}{c}$

Self Practice 8.3

1. State the following vectors in terms of $\underset{\sim}{a}$.



Smart TIPS

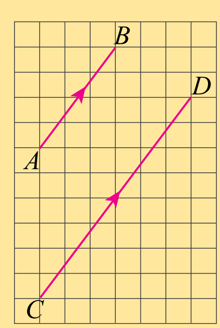
Multiplication of vector with scalar will also produce vector quantity.
For example,
$F = ma$.

Force (vector)
= mass (scalar) × acceleration (vector)

Making and verifying conjectures about the parallel vectors

INQUIRY 1 In groups

Aim: To create and verify conjectures about the relation between two parallel vectors
Instruction:



1. Consider the diagram on the right and answer the following questions:
   (a) Find the magnitude for each vector.
   (b) Determine the ratio of $|\overrightarrow{AB}| : |\overrightarrow{CD}|$.
   (c) Determine the gradient for straight line AB and CD. Are the straight line AB and CD parallel?
   (d) Express $\overrightarrow{AB}$ in term of $\overrightarrow{CD}$.
2. Given two parallel vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, what is the relation between $\underset{\sim}{a}$ and $\underset{\sim}{b}$? Discuss among your group members.

From the results of Inquiry 1, we can conclude that if two vectors are parallel, then one vector is a product of scalar with another vector.

> $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if and only if $\underset{\sim}{a} = k\underset{\sim}{b}$, where k is a constant

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are two non-zero vectors and are not parallel, then $h\underset{\sim}{a} = k\underset{\sim}{b}$, thus $h = k = 0$.

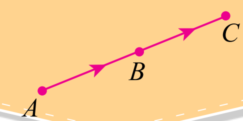
MATHEMATICS POCKET

Given three points, A, B and C. The following are the conditions for the points, to be collinear.

(a) $\overrightarrow{AB} = k\overrightarrow{BC}$.

(b) AB parallel to BC.

(c) B is common point.



Example 6

Given $\overrightarrow{PQ} = \underset{\sim}{a}$, $\overrightarrow{QR} = \underset{\sim}{b}$, $\overrightarrow{RS} = -2\underset{\sim}{a}$ and $\overrightarrow{ST} = 4\underset{\sim}{b}$. Which pairs of vectors are parallel?

Solution

Given that $\overrightarrow{PQ} = \underset{\sim}{a}$ and $\overrightarrow{RS} = -2\underset{\sim}{a}$, then $\overrightarrow{RS} = -2\overrightarrow{PQ}$. Thus, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel.

Given that $\overrightarrow{QR} = \underset{\sim}{b}$ and $\overrightarrow{ST} = 4\underset{\sim}{b}$, then $\overrightarrow{ST} = 4\overrightarrow{QR}$. Thus, $\overrightarrow{QR}$ and $\overrightarrow{ST}$ are parallel.

Example 7

Given $\overrightarrow{PQ} = \underset{\sim}{u}$ and $\overrightarrow{QR} = 5\underset{\sim}{u}$, show that P, Q and R are collinear.

Solution

Given $\overrightarrow{PQ} = \underset{\sim}{u}$ and $\overrightarrow{QR} = 5\underset{\sim}{u}$, then, $\overrightarrow{QR} = 5\overrightarrow{PQ}$.

Thus, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel.
Since Q is a common point, P, Q and R are collinear.

Mind Challenge

Given that points X, Y and Z are collinear. Write the relation between $\overrightarrow{XY}$, $\overrightarrow{XZ}$ and $\overrightarrow{YZ}$.

Example 8

Given that non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel and $(h-1)\underset{\sim}{a} = (k+5)\underset{\sim}{b}$, where h and k are constant, find the value of h and k.

Solution

Given $(h-1)\underset{\sim}{a} = (k+5)\underset{\sim}{b}$. When $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel and non-zero, then

$h - 1 = 0$ and $k + 5 = 0$

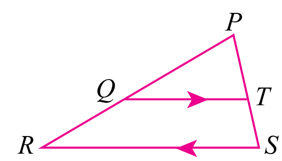
$h = 1$ and $k = -5$

8.1.4

CHAPTER 8

Self Practice 8.4

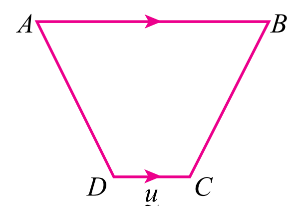
1. Given that $\overrightarrow{AB} = 5\underset{\sim}{a}$ and $\overrightarrow{PQ} = 20\underset{\sim}{a}$, express $\overrightarrow{AB}$ in terms of $\overrightarrow{PQ}$ if $\overrightarrow{AB}$ is parallel to $\overrightarrow{PQ}$.
2. Show that point L, M and N are collinear given that $\overrightarrow{LM} = 6\underset{\sim}{x}$ and $\overrightarrow{MN} = 18\underset{\sim}{x}$.
3. Given that non-zero vector, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not parallel find the value of m and n for each of the following.
   (a) $(4m + 3)\underset{\sim}{u} = (n - 7)\underset{\sim}{v}$
   (b) $(m + n - 1)\underset{\sim}{u} - (m - 2n - 10)\underset{\sim}{v} = 0$
4. Given that $\overrightarrow{XY}$ and $\overrightarrow{VW}$ are parallel vectors, $|\overrightarrow{XY}| = 6$ units and $|\overrightarrow{VW}| = 21$ units, express $\overrightarrow{VW}$ in terms of $\overrightarrow{XY}$.
5. The points P, Q and R are collinear with $\overrightarrow{PQ} = \underset{\sim}{a}$ and $\overrightarrow{QR} = (k - 2)\underset{\sim}{a}$. Find the value of k if $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{PR}$, where k is a constant.
6. In the triangle PRS, $\overrightarrow{QT}$ and $\overrightarrow{RS}$ are two parallel vectors. Given that $PT : TS = 5 : 3$, express $\overrightarrow{SR}$ in terms of $\overrightarrow{QT}$.



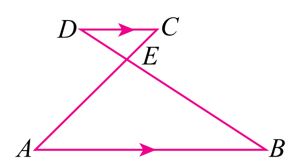
Intensive Practice 8.1

Scan the QR code or visit bit.ly/35mhddR for the quiz

1. The diagram on the right shows a trapezium $ABCD$. Given that $\overrightarrow{DC} = \underset{\sim}{u}$, $AB = 6$ cm and $DC = 2$ cm, write $\overrightarrow{AB}$ in terms of $\underset{\sim}{u}$.



2. In the diagram on the right, AB and DC are parallel. Given that $\overrightarrow{DC} = \frac{1}{3}\overrightarrow{AB}$ and $|\overrightarrow{DC}| = 4$ cm.
   (a) Find $|\overrightarrow{AB}|$.
   (b) If $\overrightarrow{AE} = 6\underset{\sim}{a}$ and $\overrightarrow{ED} = 2\underset{\sim}{b}$, express
      (i) $\overrightarrow{EC}$ in terms of $\underset{\sim}{a}$,
      (ii) $\overrightarrow{BE}$ in terms of $\underset{\sim}{b}$.



3. Given that $\overrightarrow{AB} = 4\underset{\sim}{x}$ and $\overrightarrow{AC} = 6\underset{\sim}{x}$, show that A, B and C are collinear.
4. Vector $\underset{\sim}{a}$ and vector $\underset{\sim}{b}$ are non-zero and not parallel. Given that $(h + k)\underset{\sim}{a} = (h - k + 1)\underset{\sim}{b}$ where h and k are constant. Find the value of h and k.
5. Given that $\overrightarrow{PQ} = (k + 2)\,\underset{\sim}{x} + 4\underset{\sim}{y}$. If PQ is extended to point R with $\overrightarrow{QR} = h\underset{\sim}{x} + \underset{\sim}{y}$, express k in terms of h.

8.2 Addition and Subtraction of Vectors

Performing addition and subtraction of vectors to produce resultant vectors

INQUIRY 2 In pairs 21st Century Learning

Aim: To identify the resultant vector

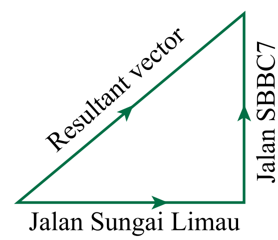
Instruction:

1. Observe the map on the right.
2. Dayang, Mia, Tan and Ranjit decided to meet at the mini market.
3. Sketch the paths that can be taken by them. Include the starting point and the terminal point together with the directions.
4. What can you say about the paths taken by them?

From the results of Inquiry 2, it is found that the sketch of the path taken by them produced a displacement, which is a resultant vector. Resultant vector is the single vector that is produced from addition of a few vectors.

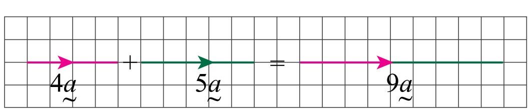
The following are a few cases that involve resultant vector.

Sketch of Ranjit's path



Case 1 **Addition and subtraction of parallel vector**

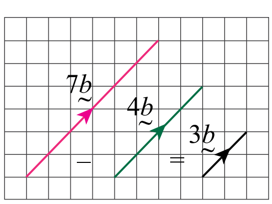
A Addition of two parallel vector



$4\underset{\sim}{a} + 5\underset{\sim}{a} = 9\underset{\sim}{a}$

$|9\underset{\sim}{a}| = |4\underset{\sim}{a}| + |5\underset{\sim}{a}|$

B Subtraction of two parallel vector



$7\underset{\sim}{b} - 4\underset{\sim}{b} = 7\underset{\sim}{b} + (-4\underset{\sim}{b}) = 3\underset{\sim}{b}$

$|3\underset{\sim}{b}| = |7\underset{\sim}{b}| - |4\underset{\sim}{b}|$

If vector $\underset{\sim}{a}$ is parallel with vector $\underset{\sim}{b}$, then $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$.

8.2.1

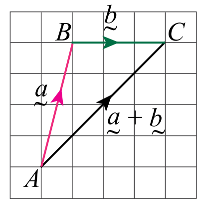
CHAPTER 8

Case 2 Addition and subtraction of non-parallel vector

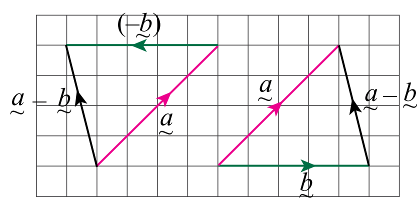
> **BRAINSTORMING**
> Does addition of vectors follows commutative laws? Discuss.

A Triangle Law

Triangle law for addition of two non-parallel vectors is given as $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

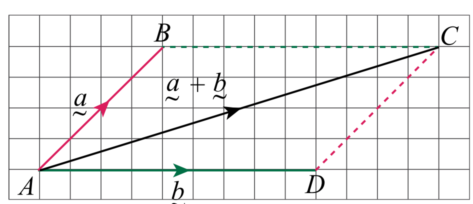


This triangle law can be used for subtraction of two non-parallel vectors.



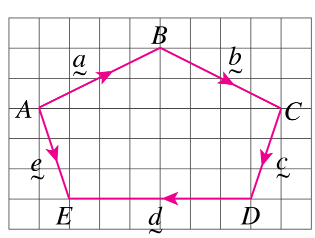
B Parallelogram Law

Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ initiated from the same point can be represented by two sides of a parallelogram, $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Thus, the resultant vector of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is the diagonal of parallelogram, $\overrightarrow{AC}$.



C Polygon Law

Polygon law is given by $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$.



Example 9

The diagram on the right shows a parallelogram *PQRS*.

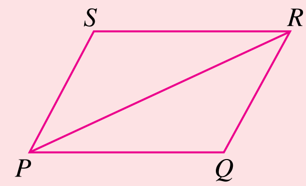
(a) Express

(i) $\overrightarrow{PQ}$ in terms of $\overrightarrow{PS}$ and $\overrightarrow{SQ}$,

(ii) $\overrightarrow{PR}$ in terms of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$,

(iii) $\overrightarrow{QR}$ in terms of $\overrightarrow{PR}$ and $\overrightarrow{PQ}$.

(b) Given that $\overrightarrow{PQ} = 2\underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{PS} = 2\underset{\sim}{b} - \underset{\sim}{a}$, express $\overrightarrow{PR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Solution

(a) (i) $\overrightarrow{PQ} = \overrightarrow{PS} + \overrightarrow{SQ}$ ← Triangle Law

(ii) $\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{PQ}$ ← Parallelogram Law

(iii) $\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$ ← Triangle law

$= -\overrightarrow{PQ} + \overrightarrow{PR}$

$= \overrightarrow{PR} - \overrightarrow{PQ}$

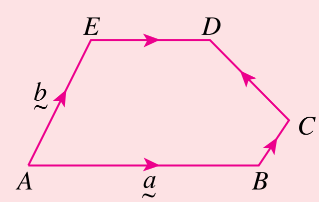
(b) $\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{PQ}$

$= 2\underset{\sim}{b} - \underset{\sim}{a} + 2\underset{\sim}{a} + \underset{\sim}{b}$

$= \underset{\sim}{a} + 3\underset{\sim}{b}$

Example 10

The diagram on the right shows a pentagon *ABCDE*. Given that $\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AE}$, $\overrightarrow{ED} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AE} = \underset{\sim}{b}$, express $\overrightarrow{CD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Solution

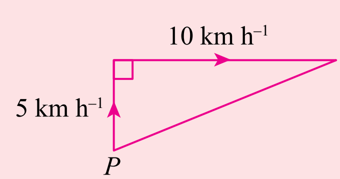
$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{ED}$

$= -\frac{1}{3}\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$= \frac{2}{3}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

Example 11

Hamzah rows his boat from point *P* across the river with velocity, $\underset{\sim}{v}$, 5 km h^{-1} to the north. The river stream flows with velocity, $\underset{\sim}{a}$, 10 km h^{-1} to the east. The diagram on the right shows the sketch movement of the boat and the river stream. Calculate the new direction and velocity of the boat after affected by the river stream.

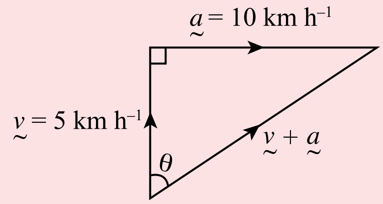


CHAPTER 8

Solution

Real velocity of the boat is $\underset{\sim}{v} + \underset{\sim}{a}$.

$|\underset{\sim}{v} + \underset{\sim}{a}| = \sqrt{5^2 + 10^2}$
$= 11.18 \text{ km h}^{-1}$



If θ is the angle formed by the north direction,

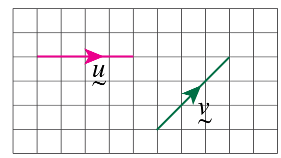
then, $\tan \theta = \dfrac{10}{5}$

$\theta = 63.43°$

The boat is moving at the bearing 063.43° with the velocity of 11.18 km h^{-1}.

Self Practice 8.5

1. The diagram on the right shows vector $\underset{\sim}{u}$ and vector $\underset{\sim}{v}$.
Draw and label the resultant vector for each of the following:

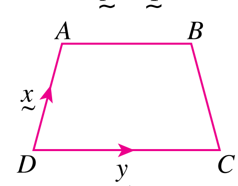


(a) $2\underset{\sim}{u} + \underset{\sim}{v}$ (b) $\dfrac{1}{2}\underset{\sim}{v} + 2\underset{\sim}{u}$

(c) $\underset{\sim}{u} - 2\underset{\sim}{v}$ (d) $2\underset{\sim}{u} - \dfrac{3}{2}\underset{\sim}{v}$

2. Vector $\underset{\sim}{p}$ represents the velocity of 70 km h^{-1} to the south and vector $\underset{\sim}{q}$ represent the velocity of 80 km h^{-1} to the east. Find the direction and magnitude of the resultant vector, $\underset{\sim}{p} + \underset{\sim}{q}$.

3. Given that $ABCD$ is a trapezium with $3AB = 2DC$.
Express the following in terms of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.



(a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$

(c) $\overrightarrow{BC}$ (d) $\overrightarrow{BD}$

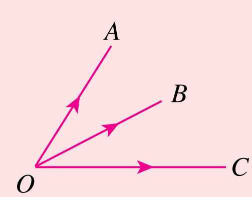
4. An airplane is flying to the north from airport P to airport Q for 1 200 km in 2 hours. The wind blows from west with the velocity of 160 km h^{-1}. Find
(a) the velocity of the plane without the influence of the wind,
(b) the original direction of the plane.

Solving problem involving vector

The problem involving addition and subtraction of vector for parallel vectors and non-parallel vectors can be solved using triangle law, parallelogram law and polygon law.

Example 12 MATHEMATICS APPLICATION

The position vector for three toy cars, A, B and C are $\overrightarrow{OA} = \underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{OB} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$ and $\overrightarrow{OC} = h\underset{\sim}{a} + 7\underset{\sim}{b}$, where h is a constant. Find the value of h where these toy car A, B and C is placed in a straight line.



CHAPTER 8

8.2.1 8.2.2

Solution

1. Understanding the problem

- Given $\overrightarrow{OA} = \underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{OB} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$ and $\overrightarrow{OC} = h\underset{\sim}{a} + 7\underset{\sim}{b}$.
- Toy cars A, B and C are located on a straight line, thus $\overrightarrow{AC} = k\overrightarrow{AB}$ where k is a constant.
- Find the value of k and h.

2. Planning strategy

- Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ using triangle law.
- Write the relation between $\overrightarrow{AC} = k\overrightarrow{AB}$.
- Find the value of k and h by comparing the coefficient in the relation $\overrightarrow{AC} = k\overrightarrow{AB}$.

3. Implementing the strategy

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -\underset{\sim}{a} - \underset{\sim}{b} + 3\underset{\sim}{a} - 2\underset{\sim}{b}\\ &= 2\underset{\sim}{a} - 3\underset{\sim}{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= -\overrightarrow{OA} + \overrightarrow{OC}\\ &= -\underset{\sim}{a} - \underset{\sim}{b} + h\underset{\sim}{a} + 7\underset{\sim}{b}\\ &= (h-1)\underset{\sim}{a} + 6\underset{\sim}{b}\end{aligned}$$

$$\overrightarrow{AC} = k\overrightarrow{AB}$$

$(h-1)\underset{\sim}{a} + 6\underset{\sim}{b} = k(2\underset{\sim}{a} - 3\underset{\sim}{b})$

$(h-1)\underset{\sim}{a} + 6\underset{\sim}{b} = (2k)\underset{\sim}{a} - (3k)\underset{\sim}{b}$

Compare the coefficient of $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$h - 1 = 2k$ and $6 = -3k$

$h = 2k + 1$ $\quad k = -2$

Substitute $k = -2$ into $h = 2k + 1$,

$h = 2(-2) + 1$

$= -3$

4. Making a conclusion

When $k = -2$,

$$\begin{aligned}\overrightarrow{AC} &= k\overrightarrow{AB}\\ &= (-2)(2\underset{\sim}{a} - 3\underset{\sim}{b})\\ &= -4\underset{\sim}{a} + 6\underset{\sim}{b}\end{aligned}$$

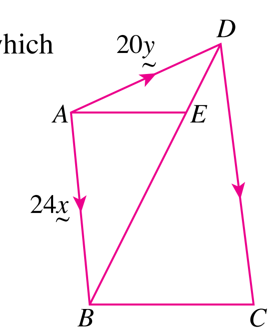
When $h = -3$,

$$\begin{aligned}\overrightarrow{AC} &= (h-1)\underset{\sim}{a} + 6\underset{\sim}{b}\\ &= (-3-1)\underset{\sim}{a} + 6\underset{\sim}{b}\\ &= -4\underset{\sim}{a} + 6\underset{\sim}{b}\end{aligned}$$

Self Practice 8.6

1. Given O, X, Y and Z are four points with $\overrightarrow{OX} = 4\underset{\sim}{x} - 2\underset{\sim}{y}$, $\overrightarrow{OY} = k\underset{\sim}{x} - \underset{\sim}{y}$ and $\overrightarrow{OZ} = 6\underset{\sim}{x} + 5\underset{\sim}{y}$. If points X, Y and Z are collinear, find the value of k.

2. The diagram on the right shows the plan of alleys of a residential area which forms a quadrilateral $ABCD$. There is a lamp post at position E, where $BE : ED = 3 : 1$. Alley AB and DC are parallel and $DC = \frac{4}{3}AB$.



   (a) Express $\overrightarrow{BD}$ and $\overrightarrow{AE}$ in terms of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

   (b) Show that the alley AE is parallel to alley BC.

8.2.2

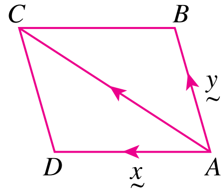
CHAPTER 8

Intensive Practice 8.2

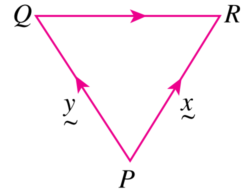
Scan the QR code or visit bit.ly/2pWaGpX for the quiz

1. Express the following vectors in terms of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

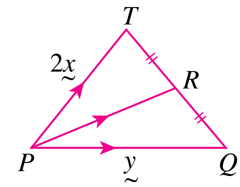
(a) $\overrightarrow{AC}$



(b) $\overrightarrow{QR}$

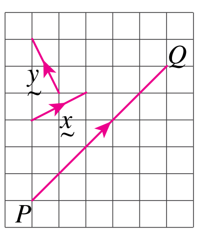


(c) $\overrightarrow{PR}$

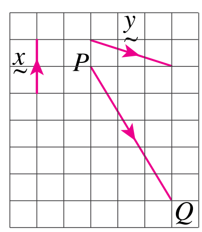


2. For the following diagrams, express vector $\overrightarrow{PQ}$ in terms of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

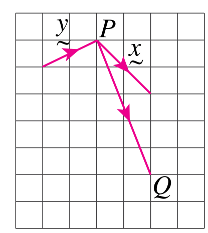
(a)



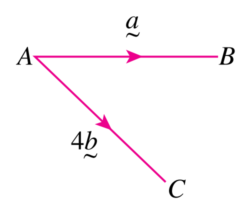
(b)



(c)

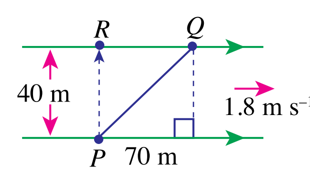


3. In the diagram on the right, $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = 4\underset{\sim}{b}$. Given Q is a point on AC where $AQ : QC = 1 : 3$. Express $\overrightarrow{BQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

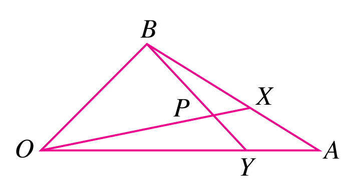


4. Given that $\underset{\sim}{p} = 2\underset{\sim}{a} + 3\underset{\sim}{b}$, $\underset{\sim}{q} = 4\underset{\sim}{a} - \underset{\sim}{b}$ and $\underset{\sim}{r} = h\underset{\sim}{a} + (h + k)\underset{\sim}{b}$ where h and k are constants. Find the value of h and k if $\underset{\sim}{r} = 3\underset{\sim}{p} - 4\underset{\sim}{q}$.

5. The diagram on the right shows the sketch of a river. The width of the river is 40 m and the velocity of the downstream river flow is 1.8 m s^{-1}. Hamid wanted to row his boat from P across the river at R, but his boat was swept by the current flow and stopped at Q in 12 seconds. Calculate the speed of Hamid's boat.



6. The diagram on the right shows a triangle OAB.
Given that $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $5BX = 3BA$ and $OY : OA = 3 : 4$.



(a) Find the following in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(i) $\overrightarrow{BA}$ (ii) $\overrightarrow{BX}$

(iii) $\overrightarrow{OX}$ (iv) $\overrightarrow{BY}$

(b) Given that $\overrightarrow{OP} = \lambda\overrightarrow{OX}$ and $\overrightarrow{BP} = \mu\overrightarrow{BY}$. Express $\overrightarrow{OP}$ in terms of

(i) λ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, (ii) μ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

(c) Hence, find the value of λ and μ.

8.3 Vectors in Cartesian Plane

Representing vectors and determining the magnitude of vectors in the Cartesian plane

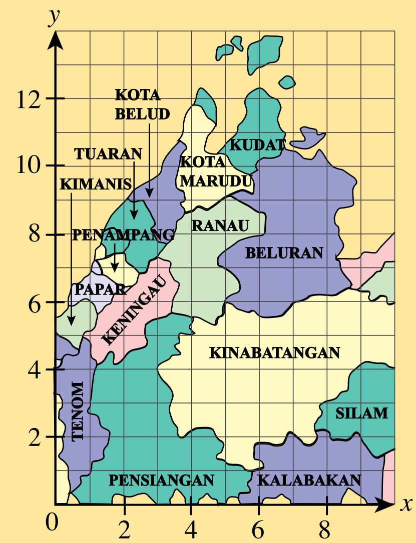
INQUIRY 3 In groups 21st Century Learning

Aim: To identify the resultant vector

Instruction:

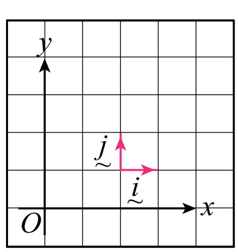
1. Observe the map of Sabah that is drawn on a Cartesian plane on the right.
2. Observe the following situation carefully:

> Arding wishes to explore a province in Sabah. Arding is at a place located at coordinate (1, 3). Next, he moves 5 units parallel to x-axis and 4 units parallel to y-axis to a location in another province. He promises to meet his friend, Timan at the location. Timan moves at translation $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ from his place to meet Arding.



3. Plot on the Cartesian plane, the movement and the position of Arding and Timan.
4. What is the name of the province where they both meet?
5. Express the translation of the movement of Arding from the location of the first province to the location of the second province.
6. Find the distance, in units, between Arding's first location and Timan's first location with their meeting point.
7. Present the findings in front of the class and host a question and answer session with your friends.

From the results of Inquiry 3, a vector can be expressed as the combination of parallel vector and non-parallel vector. On the Cartesian plane, vector will be expressed as the combination of parallel vector with x-axis and y-axis.



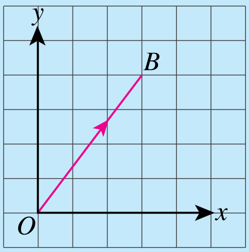
Vector with magnitude 1 unit and parallel with x-axis is called vector $\underset{\sim}{i}$ and is written as $\underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\underset{\sim}{i}| = 1$.

Vector with magnitude 1 unit and parallel with y-axis is called vector $\underset{\sim}{j}$ and is written as $\underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|\underset{\sim}{j}| = 1$.

CHAPTER 8

8.3.1

Observe the following diagram:



- The coordinates of point B is $B(x, y)$.
- The position vector of point B relative to point O is $\overrightarrow{OB}$.
- $\overrightarrow{OB}$ can be written as the combination of vector $\underset{\sim}{i}$ and $\underset{\sim}{j}$, which is $x\underset{\sim}{i} + y\underset{\sim}{j}$.
- $\overrightarrow{OB}$ can be written in the form of column vector, $\begin{pmatrix} x \\ y \end{pmatrix}$.
- Magnitude of $\overrightarrow{OB} = \sqrt{x^2 + y^2}$

Example 13

Given that points $A(1, 2)$, $B(-4, 5)$, $C(8, -3)$, $D(-7, -4)$ and O is the origin on a Cartesian plane. Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ in the forms of

(a) $\begin{pmatrix} x \\ y \end{pmatrix}$ (b) $x\underset{\sim}{i} + y\underset{\sim}{j}$

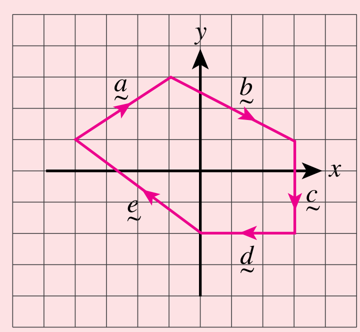
Solution

(a) $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$, $\overrightarrow{OD} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$

(b) $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{OB} = -4\underset{\sim}{i} + 5\underset{\sim}{j}$, $\overrightarrow{OC} = 8\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{OD} = -7\underset{\sim}{i} - 4\underset{\sim}{j}$

Example 14

The diagram on the right shows vector $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ and $\underset{\sim}{e}$ on a Cartesian plane.



(a) Express each vector in the forms of $x\underset{\sim}{i} + y\underset{\sim}{j}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$.
(b) Find the magnitude for every vector.
(c) Are vectors $\underset{\sim}{b}$ and $\underset{\sim}{e}$ parallel? Give your reason.

Solution

(a) $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\underset{\sim}{c} = -3\underset{\sim}{j}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$, $\underset{\sim}{d} = -3\underset{\sim}{i}$, $\underset{\sim}{d} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$\underset{\sim}{e} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{e} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

CHAPTER 8

(b) $|\underset{\sim}{a}| = \sqrt{3^2 + 2^2} = 3.606$ units , $|\underset{\sim}{b}| = \sqrt{4^2 + (-2)^2} = 4.472$ units

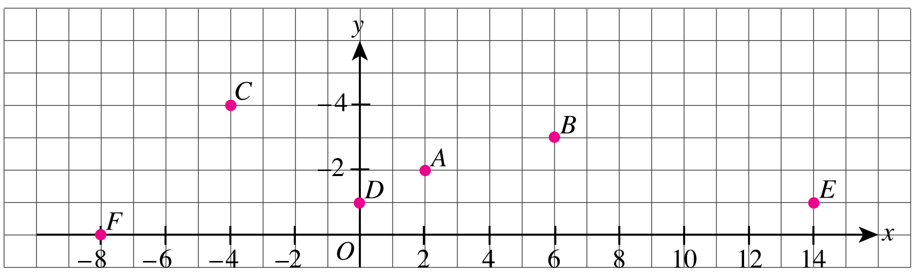
$|\underset{\sim}{c}| = \sqrt{0^2 + (-3)^2} = 3$ units , $|\underset{\sim}{d}| = \sqrt{(-3)^2 + 0^2} = 3$ units

$|\underset{\sim}{e}| = \sqrt{(-4)^2 + 3^2} = 5$ units

(c) Vectors $\underset{\sim}{b}$ and $\underset{\sim}{e}$ are not parallel because $\underset{\sim}{b} \neq k\underset{\sim}{e}$ or the gradient $\underset{\sim}{b} \neq$ gradient $\underset{\sim}{e}$.

Self Practice 8.7

1. The diagram below shows 6 points on a Cartesian plane.



Express $\overrightarrow{OA}$, $\overrightarrow{OF}$, $\overrightarrow{BC}$, $\overrightarrow{FA}$, $\overrightarrow{DE}$ and $\overrightarrow{DO}$ in the form of

(a) $x\underset{\sim}{i} + y\underset{\sim}{j}$,

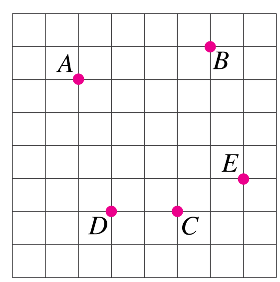
(b) column vector.

2. Given that point $A(-2, 3)$, point $B(5, 8)$ and O which is the origin on a Cartesian plan.

(a) Find the position vector of point B.

(b) Calculate $|\overrightarrow{AB}|$.

3. The diagram on the right shows 5 points, A, B, C, D and E on a grid.



(a) Express the following vectors in the form of resultant vector of vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{BA}$

(iii) $\overrightarrow{BC}$ (iv) $\overrightarrow{DC}$

(v) $\overrightarrow{AC}$ (vi) $\overrightarrow{DE}$

(b) State the pair of vectors that are parallel and explain your reason.

(c) State the pair of vectors that are negative and give your reason.

4. Given that $\underset{\sim}{p} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\underset{\sim}{q} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ are representing the position vector of point P, Q and R.

(a) Write the vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ in the form of $x\underset{\sim}{i} + y\underset{\sim}{j}$.

(b) State the coordinates of points P, Q and R.

(c) Calculate the length of vector $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$.

Describing and determining the unit vector in a direction of a vector

You have learned that $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors which are parallel to positive x-axis and y-axis respectively. Let's find out about the unit vector in the direction of vector that is not parallel to the x-axis or y-axis.

INQUIRY 4 In pairs | 21st Century Learning

Aim: To determine the unit vectors in the direction of a vector given

Instruction:

1. Scan the QR code or visit the link on the right.
2. Drag the slider x_1 and y_1 to view the changes of the unit vector on Cartesian plane and the calculated unit vector.
3. Compare the unit vector with each of the changes in x_1 and y_1 values.
4. Discuss the formula used to find the unit vector in direction with a vector.

bit.ly/2Is5lNN

From the results of Inquiry 4, unit vector in the direction of a vector can be found by dividing vector with the magnitude of the vector.

In general:

If $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then vector unit in direction $\underset{\sim}{r}$ is

$$\hat{\underset{\sim}{r}} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j}}{\sqrt{x^2 + y^2}}.$$

MATHEMATICS POCKET

Unit vector is a vector in the direction of a vector with magnitude of 1 unit.

Example 15

Given point $A(4, 3)$, find the unit vector in the direction of vector $\overrightarrow{OA}$. Express the answer in the forms of

(a) component $\underset{\sim}{i}$ and $\underset{\sim}{j}$, (b) column vector.

Solution

(a) $\overrightarrow{OA} = \underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$

$|\underset{\sim}{a}| = \sqrt{4^2 + 3^2}$

$= 5$ units

Unit vector in the component $\underset{\sim}{i}$ and $\underset{\sim}{j}$ is $\hat{\underset{\sim}{a}} = \dfrac{4\underset{\sim}{i} + 3\underset{\sim}{j}}{5}$.

(b) Unit vector in the form of column vector is

$$\hat{\underset{\sim}{a}} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \text{ or } \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$$

QUICK COUNT

Determine the magnitude of vector $4\underset{\sim}{i} + 3\underset{\sim}{j}$ using scientific calculator.

1. Press MENU
2. Press 1
3. Press SHIFT +
4. Screen will show:

   Pol (

5. Press 4 SHIFT) 3 =
6. Screen will show:

   Pol (4, 3)
   r = 5

CHAPTER 8

Example 16

Given that $-\frac{1}{3}\underset{\sim}{i} + k\underset{\sim}{j}$ is a unit vector, find the value of k.

Solution

$$\sqrt{\left(-\frac{1}{3}\right)^2 + k^2} = 1 \quad \leftarrow \text{Magnitude of the unit vector is 1}$$
$$\sqrt{\frac{1}{9} + k^2} = 1$$
$$\frac{1}{9} + k^2 = 1$$
$$k^2 = \frac{8}{9}$$
$$k = \pm 0.9428$$

Self Practice 8.8

1. Calculate the magnitude for the following vectors.

(a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} -4 \\ -7 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ -\frac{4}{7} \end{pmatrix}$ (d) $-12\underset{\sim}{i} - 5\underset{\sim}{j}$ (e) $6\underset{\sim}{i}$

2. Find the unit vector in the direction of the following vectors.

(a) $3\underset{\sim}{i} + 2\underset{\sim}{j}$ (b) $-\underset{\sim}{i} - 9\underset{\sim}{j}$ (c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} -8 \\ -15 \end{pmatrix}$

3. Determine whether the following vectors are unit vectors.

(a) $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} -0.6 \\ -0.8 \end{pmatrix}$ (d) $\frac{7}{25}\underset{\sim}{i} + \frac{24}{25}\underset{\sim}{j}$ (e) $\frac{2}{3}\underset{\sim}{i} + \frac{\sqrt{7}}{3}\underset{\sim}{j}$

4. Find the value of k for the following unit vectors.

(a) $\begin{pmatrix} 0 \\ k \end{pmatrix}$ (b) $\begin{pmatrix} k \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} k \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} k \\ k \end{pmatrix}$ (e) $0.5\underset{\sim}{i} + k\underset{\sim}{j}$ (f) $k\underset{\sim}{i} + \frac{13}{84}\underset{\sim}{j}$

5. Given the unit vector in the direction of vector $\underset{\sim}{u}$ is $\hat{\underset{\sim}{u}} = \frac{p\underset{\sim}{i} + 8\underset{\sim}{j}}{\sqrt{73}}$, find the possible values of p.

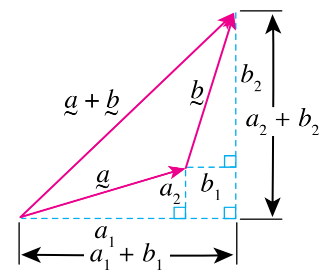
6. Given $\hat{\underset{\sim}{u}} = (1 - k)\underset{\sim}{i} + h\underset{\sim}{j}$, express h in terms of k.

Performing arithmetic operations on two or more vectors

A Addition of two or more vectors

Consider $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$



Then, $\underset{\sim}{a} + \underset{\sim}{b} = (a_1\underset{\sim}{i} + a_2\underset{\sim}{j}) + (b_1\underset{\sim}{i} + b_2\underset{\sim}{j})$
$= (a_1 + b_1)\underset{\sim}{i} + (a_2 + b_2)\underset{\sim}{j}$ ← Gather component $\underset{\sim}{i}$ and $\underset{\sim}{j}$, then sum up separately

Example 17

Find the addition of the following vectors.

(a) $\underset{\sim}{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

(b) $\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{w} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$

Solution

(a) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(b) $\underset{\sim}{v} + \underset{\sim}{w} = (3\underset{\sim}{i} + 2\underset{\sim}{j}) + (4\underset{\sim}{i} - 5\underset{\sim}{j})$
$= (3 + 4)\underset{\sim}{i} + (2 - 5)\underset{\sim}{j}$
$= 7\underset{\sim}{i} - 3\underset{\sim}{j}$

B Subtraction of two vectors

The same method for the addition of vectors can be used in the operation of subtraction of two vectors.

Example 18

Find $\underset{\sim}{p} - \underset{\sim}{q}$ for the following pairs of vectors.

(a) $\underset{\sim}{p} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(b) $\underset{\sim}{p} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{q} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$

Solution

(a) $\underset{\sim}{p} - \underset{\sim}{q} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 7 - 4 \\ -1 - 1 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(b) $\underset{\sim}{p} - \underset{\sim}{q} = (2\underset{\sim}{i} - \underset{\sim}{j}) - (3\underset{\sim}{i} + 5\underset{\sim}{j})$
$= (2 - 3)\underset{\sim}{i} + (-1 - 5)\underset{\sim}{j}$
$= -\underset{\sim}{i} - 6\underset{\sim}{j}$

C Multiplication of vector with scalar

When a vector is multiplied with a scalar, both component $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are also multiplied with the scalar.

Example 19

For each of the following vectors, find

(a) $-3\underset{\sim}{s}$, given $\underset{\sim}{s} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$,

(b) $2\underset{\sim}{r}$, given $\underset{\sim}{r} = 5\underset{\sim}{i} - 3\underset{\sim}{j}$.

Solution

(a) $-3\underset{\sim}{s} = -3\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 12 \\ -6 \end{pmatrix}$

(b) $2\underset{\sim}{r} = 2(5\underset{\sim}{i} - 3\underset{\sim}{j})$

$= 10\underset{\sim}{i} - 6\underset{\sim}{j}$

Smart TIPS

Arithmetic operation involving **parallel vector** is performed using the same method as **non-parallel vector**.

D Combination of arithmetic operation on vectors

Combination of arithmetic operation applied on vectors need to follow the operation rules of mathematics. Multiplication with scalar need to be performed before addition and subtraction.

Example 20

Given $\underset{\sim}{p} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\underset{\sim}{q} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, determine the vector $3\underset{\sim}{p} + \underset{\sim}{q} - 2\underset{\sim}{r}$.

Solution

$$3\underset{\sim}{p} + \underset{\sim}{q} - 2\underset{\sim}{r} = 3\begin{pmatrix} 6 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ -9 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix} - \begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -20 \end{pmatrix}$$

Self Practice 8.9

1. Given $\underset{\sim}{a} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ -12 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$, find

(a) $2\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}$ (b) $-3\underset{\sim}{a} + 2\underset{\sim}{b} - \underset{\sim}{c}$ (c) $\frac{1}{2}\underset{\sim}{b} + \underset{\sim}{c} - 3\underset{\sim}{a}$ (d) $\frac{1}{4}\underset{\sim}{b} - \underset{\sim}{a} + 3\underset{\sim}{c}$

2. Given $\underset{\sim}{u} = 3\underset{\sim}{i} + 6\underset{\sim}{j}$, $\underset{\sim}{v} = -2\underset{\sim}{i} - 8\underset{\sim}{j}$ and $\underset{\sim}{w} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$, find

(a) $\underset{\sim}{u} - 2\underset{\sim}{v} + \underset{\sim}{w}$ (b) $3\underset{\sim}{u} + 2\underset{\sim}{v} - \underset{\sim}{w}$ (c) $\frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w} - 3\underset{\sim}{u}$ (d) $\frac{1}{4}\underset{\sim}{v} - \underset{\sim}{w} + 3\underset{\sim}{u}$

Solving problems involving vectors

By applying the knowledge learnt, problems involving vectors can be solved easily, especially the problem involving our daily lives.

Example 21 MATHEMATICS APPLICATION

One particle is moving from point $A(5, 10)$ with the velocity vector $(3\underset{\sim}{i} - \underset{\sim}{j})$ m s^{-1}. After t seconds leaving A, the particle is on point S, with $\overrightarrow{OS} = \overrightarrow{OA} + t\underset{\sim}{v}$. Find the speed and the position of the particle from O after 4 seconds. When will the particle reside on the right side of origin O?

Solution

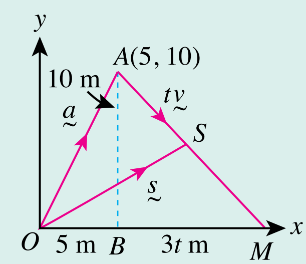
1. Understanding the problem

- Original position vector,
  $\overrightarrow{OA} = \underset{\sim}{a} = 5\underset{\sim}{i} + 10\underset{\sim}{j} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$.
- Velocity vector, $\underset{\sim}{v} = 3\underset{\sim}{i} - \underset{\sim}{j} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
- Speed is the magnitude of velocity vector.
- The particle is on the right side of O if component $\underset{\sim}{j}$ in the position vector is zero.

2. Planning the strategy

- Find $|\underset{\sim}{v}|$ to determine the speed.
- Find position of particle after 4 seconds using $\overrightarrow{OS} = \overrightarrow{OA} + t\underset{\sim}{v}$ or $\underset{\sim}{s} = \underset{\sim}{a} + \underset{\sim}{v}t$ when $t = 4$.
- The particle is on the right side of O when the y component in $\underset{\sim}{s} = \begin{pmatrix} x \\ y \end{pmatrix}$ is zero.

3. Implementing the strategy



Speed, $|\underset{\sim}{v}| = \sqrt{3^2 + (-1)^2}$

$= \sqrt{10}$ m s^{-1}

After 4 seconds, $\underset{\sim}{s} = \underset{\sim}{a} + 4\underset{\sim}{v}$,

$s = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 17 \\ 6 \end{pmatrix}$

The particle is at point (17, 6).

Position vector after t seconds,

$\underset{\sim}{s} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 5 + 3t \\ 10 - t \end{pmatrix}$

The position of the particle after t seconds is

$\overrightarrow{OS} = \underset{\sim}{s} = \begin{pmatrix} 5 + 3t \\ 10 - t \end{pmatrix}$.

The particle is at the right side of origin O when

$y = 0$

$10 - t = 0$

$t = 10$ seconds

4. Making a conclusion

Distance $AM = \sqrt{30^2 + 10^2}$

$= \sqrt{1\,000}$ m

Then, speed $= \dfrac{\sqrt{1\,000}}{10}$

$= \sqrt{10}$ m s^{-1}

Self Practice 8.10

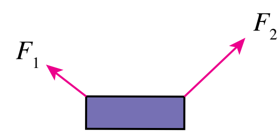
1. A toy car is at point $A(-3, -2)$. The car is then moved with a constant velocity $(2\underset{\sim}{i} - 3\underset{\sim}{j})$ cm s^{-1}. Find the position vector of the toy car after 2.5 seconds.

2. The position vector of boat A, t hours after leaving the port O is $t\begin{pmatrix}30\\15\end{pmatrix}$ while the position vector of boat B is $\begin{pmatrix}50\\5\end{pmatrix} + t\begin{pmatrix}10\\10\end{pmatrix}$. Determine the velocity of boat A and boat B. Will the two boats meet?

Intensive Practice 8.3

Scan the QR code or visit bit.ly/2MqRicw for the quiz

1. Two forces $F_1 = \begin{pmatrix}-4\\3\end{pmatrix}$ and $F_2 = \begin{pmatrix}7\\5\end{pmatrix}$ are applied on an object as the diagram on the right.
   (a) Find the resultant force.
   (b) Calculate the magnitude of the resultant force.



2. Given $\underset{\sim}{p} = (k - 3)\underset{\sim}{i} + 14\underset{\sim}{j}$ and $\underset{\sim}{q} = \underset{\sim}{i} + (k - 8)\underset{\sim}{j}$ with k is a constant. If $\underset{\sim}{p}$ is parallel to $\underset{\sim}{q}$, find the value of k.

3. Given $\underset{\sim}{u} = \underset{\sim}{b} - \underset{\sim}{a}$ and $\underset{\sim}{v} = \underset{\sim}{c} - \underset{\sim}{b}$, with $\underset{\sim}{a} = \begin{pmatrix}3\\1\end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix}5\\-2\end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix}m\\-6\end{pmatrix}$. If $\underset{\sim}{u}$ is parallel to $\underset{\sim}{v}$, find the value of m. After that find $|\underset{\sim}{u}| : |\underset{\sim}{v}|$.

4. Given the triangle ABC with $\overrightarrow{AB} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\overrightarrow{AC} = 10\underset{\sim}{i} + 5\underset{\sim}{j}$. R is a point on BC with $\overrightarrow{BR} = \frac{1}{2}\overrightarrow{BC}$. Find
   (a) $\overrightarrow{BC}$,
   (b) the unit vector in the direction of $\overrightarrow{BC}$,
   (c) $\overrightarrow{AR}$.

5. A swimmer swims at a velocity $\underset{\sim}{v} = \begin{pmatrix}2.4\\1.5\end{pmatrix}$. It is found that the stream flows at a velocity of $\underset{\sim}{a} = \begin{pmatrix}0.5\\-2.1\end{pmatrix}$. Find the magnitude and direction of the resultant velocity of the swimmer.

6. Given $\underset{\sim}{r} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{s} = m\underset{\sim}{i} - 3\underset{\sim}{j}$, find the value of m if
   (a) $|\underset{\sim}{r} + \underset{\sim}{s}| = 10$,
   (b) $\underset{\sim}{r}$ parallel to $\underset{\sim}{s}$,
   (c) $(2\underset{\sim}{r} - \underset{\sim}{s})$ is parallel to the y-axis.

7. Given $\begin{pmatrix}k\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ is a unit vector, find the value of k.

8. The length of vector $\underset{\sim}{v}$ is 5 units and the direction is opposite with vector $\begin{pmatrix}2\\-1\end{pmatrix}$, find vector $\underset{\sim}{v}$.

8.3.4

9. Vector $\underset{\sim}{p} = (m - 1)\underset{\sim}{i} + 2\underset{\sim}{j}$ is orthogonal with vector $\underset{\sim}{q} = 8\underset{\sim}{i} + n\underset{\sim}{j}$. Express m in terms of n.

10. Ship M left port O when the sea was calm, with the velocity of $v_M = 6\underset{\sim}{i} + 8\underset{\sim}{j}$ km h^{-1}. At the same time, ship N left port Q with velocity of $v_N = 4\underset{\sim}{i} + 4\underset{\sim}{j}$ km h^{-1}. Given that position vector of port Q is $\overrightarrow{OQ} = 50\underset{\sim}{i} + 20\underset{\sim}{j}$.
   (a) After t hours, position vector of ship M is $\overrightarrow{OM} = t(6\underset{\sim}{i} + 8\underset{\sim}{j})$. Find the position vector of ship N at that time.
   (b) Show that ship M will cross ship N and find the time when this occurs.

SUMMARY OF CHAPTER 8

VECTORS

representation
- $\overrightarrow{AB} = \underset{\sim}{a}$
- Negative vector of $\overrightarrow{AB}$ is $-\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Zero vector, $\underset{\sim}{0}$

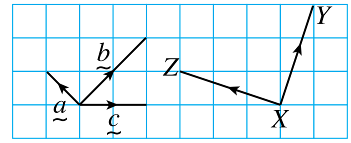
vector operation
- $k\underset{\sim}{a}$ is a vector that is parallel to $\underset{\sim}{a}$ and the magnitude is $k \times |\underset{\sim}{a}|$.
- Multiplication of vector: $k \times \underset{\sim}{a} = k\underset{\sim}{a}$
- Addition of vectors: $2\overrightarrow{AB} + 3\overrightarrow{AB} = 5\overrightarrow{AB}$
- Subtraction of vectors: $4\underset{\sim}{a} - \underset{\sim}{a} = 3\underset{\sim}{a}$

on planes
- $x\underset{\sim}{i} + y\underset{\sim}{j}$
- $\begin{pmatrix} x \\ y \end{pmatrix}$ with $\underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Unit vector in direction of $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is $\hat{\underset{\sim}{r}} = \dfrac{x\underset{\sim}{i} + y\underset{\sim}{j}}{\sqrt{x^2 + y^2}}$
- Magnitude, $|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$

WRITE YOUR JOURNAL

In pairs, find the differences between a scalar quantity and a vector quantity. Compare the methods used to perform arithmetic operations for both quantities. After that, find information from the internet regarding the application of vectors in your daily life. Write a report and discuss your findings.

MASTERY PRACTICE



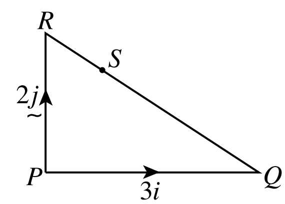
1. The diagram on the right shows three vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ that are not parallel. Express PL1
(a) $\overrightarrow{XY}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$,
(b) $\overrightarrow{XZ}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

2. Given $\overrightarrow{PQ} = 3k\underset{\sim}{a} - 4\underset{\sim}{b}$ and $\overrightarrow{XY} = 4\underset{\sim}{a} + 8\underset{\sim}{b}$. If $\overrightarrow{PQ}$ parallel to $\overrightarrow{XY}$, find the value of k. PL2

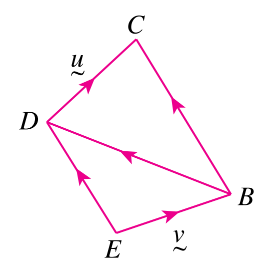
3. Given $\underset{\sim}{p} = m\underset{\sim}{i} - n\underset{\sim}{j}$ is a unit vector in the direction of $\underset{\sim}{p}$, express m in terms of n. PL2

4. Given $\underset{\sim}{u} = k\underset{\sim}{i} + h\underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - 4\underset{\sim}{j}$. If $|\underset{\sim}{u} + \underset{\sim}{v}| = \sqrt{k^2 + h^2}$, express h in terms of k. PL2

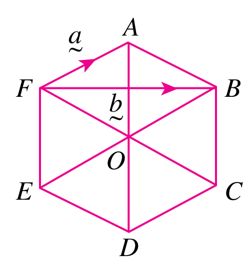
5. Given $A(3, 4)$, $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$. Find PL2
(a) the unit vector in the direction of $\overrightarrow{AC}$,
(b) the coordinates C.



6. The diagram on the right shows triangle PQR with $\overrightarrow{PQ} = 3\underset{\sim}{i}$ and $\overrightarrow{PR} = 2\underset{\sim}{j}$. Given $\overrightarrow{RS} : \overrightarrow{SQ} = 2 : 3$, express $\overrightarrow{RS}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$. PL3



7. The diagram on the right shows a trapezium $BCDE$ with $\overrightarrow{DC} = \underset{\sim}{u}$ and $\overrightarrow{EB} = \underset{\sim}{v}$. If $\overrightarrow{ED} = \frac{1}{2}\overrightarrow{BC}$, express $\overrightarrow{BC}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. PL3

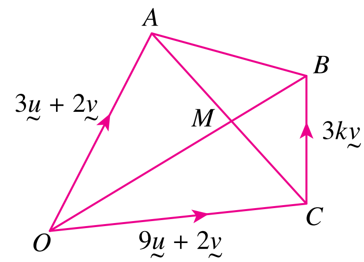


8. The diagram on the right shows a regular hexagon, $ABCDEF$ with centre O. Given $\overrightarrow{FA} = \underset{\sim}{a}$ and $\overrightarrow{FB} = \underset{\sim}{b}$, PL3
(a) express the following in terms of $\underset{\sim}{a}$ and/or $\underset{\sim}{b}$,
(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{FO}$ (iii) $\overrightarrow{FC}$
(iv) $\overrightarrow{BC}$ (v) $\overrightarrow{FD}$ (vi) $\overrightarrow{AD}$
(b) state the relationship between $\overrightarrow{AB}$ and $\overrightarrow{FC}$,
(c) determine whether $\overrightarrow{AC}$ and $\overrightarrow{FD}$ are parallel.

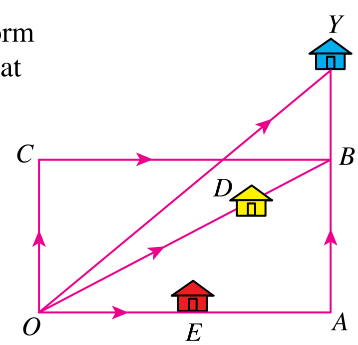
CHAPTER 8

9. The position vector of city A is $-10\underset{\sim}{i} + 10\underset{\sim}{j}$ and position vector of city B is $10\underset{\sim}{i} - 11\underset{\sim}{j}$. The position of city A, B and C are collinear and the distance between city A and city C is two times the distance between city A and city B. The distance between cities is measured in kilometer. Find PL4
   (a) vector $\overrightarrow{AB}$,
   (b) the distance between city A and city B,
   (c) vector $\overrightarrow{OC}$.

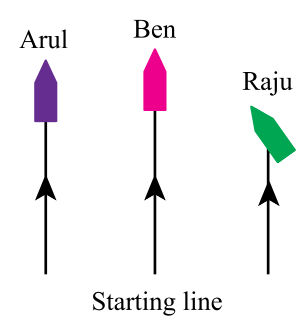
10. The diagram on the right shows a quadrilateral $OABC$. M is the midpoint of AC and $OM : OB = 2 : 3$. Given $\overrightarrow{OA} = 3\underset{\sim}{u} + 2\underset{\sim}{v}$, $\overrightarrow{OC} = 9\underset{\sim}{u} + 2\underset{\sim}{v}$ and $\overrightarrow{CB} = 3k\underset{\sim}{v}$, where k is constant, PL4
   (a) express in terms of $\underset{\sim}{u}$ and/or $\underset{\sim}{v}$,
      (i) $\overrightarrow{AC}$ (ii) $\overrightarrow{OM}$
   (b) express $\overrightarrow{OB}$ in terms of
      (i) $\underset{\sim}{u}$ and $\underset{\sim}{v}$, (ii) $\underset{\sim}{u}$, $\underset{\sim}{v}$ and k.
   Hence, find the value of k.



11. The diagram on the right shows roads of a housing area that form a rectangle $OABC$. Building D is at OB road and building E is at OA road. Given $OD = \frac{3}{4}OB$ and $OE : OA = 1 : 2$. Building Y is at AB road which is extended with $BY = \frac{1}{2}AB$. OA road is represented by vector $4\underset{\sim}{a}$ while OC road is represented by vector $4\underset{\sim}{c}$. PL5
   (a) Express vector that represents the following road in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.
      (i) $\overrightarrow{OB}$ (ii) $\overrightarrow{OD}$ (iii) $\overrightarrow{OY}$ (iv) $\overrightarrow{ED}$
   (b) Prove that building E, D and Y reside on the same straight line.

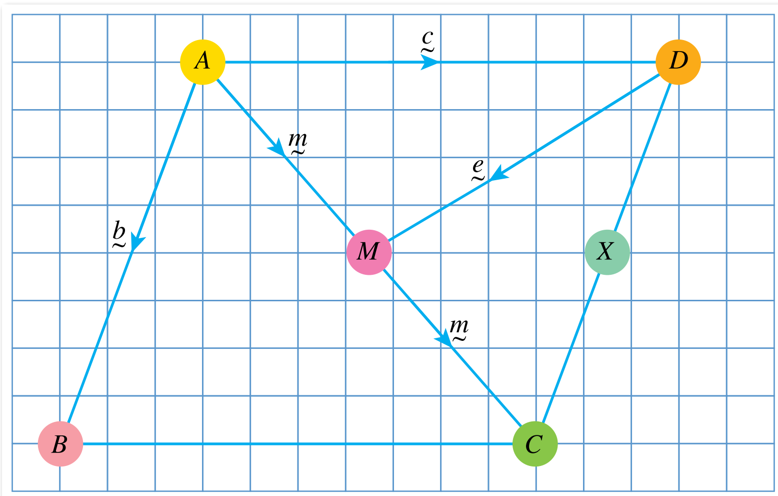


12. The diagram on the right shows the positions and directions of the boats which belong to Arul, Ben and Raju in a solar boat competition. Arul's and Ben's boats move in the same direction of the stream. The velocity of the stream is given by $\underset{\sim}{w} = \left(\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}\right)$ m s^{-1}, while the velocity of Arul's boat is $\underset{\sim}{a} = (3\underset{\sim}{i} + \underset{\sim}{j})$ m s^{-1} and the velocity of Ben's boat is $\underset{\sim}{b} = (6\underset{\sim}{i} + 2\underset{\sim}{j})$ m s^{-1}. PL5
   (a) Calculate the resultant velocity of Arul's boat and the resultant velocity of Ben's boat. After that, find the difference between the speed of the two boats.
   (b) Raju's boat is deviated from the path. Given that the velocity of Raju's boat is $\underset{\sim}{r} = \left(2\underset{\sim}{i} - \frac{4}{3}\underset{\sim}{j}\right)$ m s^{-1}. Find the unit vector in the direction of the resultant velocity of the boat.



Exploring MATHEMATICS

Madam Tan is a housewife who visits several locations every day. The diagram below shows the displacement vector $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{e}$ and $\underset{\sim}{m}$ that represent the journey of Madam Tan from her house in A to the location she normally visits.



Guide:
A : Madam Tan's house
B : Market
C : Mother's house
D : School
M : Kindergarten
X : Grocery store

Write the vectors $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{e}$ and $\underset{\sim}{m}$ in the forms of $\begin{pmatrix} x \\ y \end{pmatrix}$ and $x\underset{\sim}{i} + y\underset{\sim}{j}$. [1 side = 1 km].

1. After that, find the shortest distance from Madam Tan's house to every location according to the given displacement vector.

2. Madam Tan will send her son to the kindergarten before sending her daughter to the school. Observe that the resultant vector $\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MD} = \underset{\sim}{m} - \underset{\sim}{e}$ obeys the triangle law. State the other resultant vectors that obey the:
(a) Triangle law, (b) Parallelogram law, (c) Polygon law

3. Copy and complete the table by filling up the resultant vectors that are represented by combinations of vectors through arithmetic operations of the vectors.

	Arithmetic operation	Resultant vector		Arithmetic operation	Resultant vector		Arithmetic operation	Resultant vector
(a)	$\underset{\sim}{m} - \underset{\sim}{e}$	$\overrightarrow{AD}$	(f)	$\underset{\sim}{c} - \frac{\underset{\sim}{b}}{2}$		(k)	$\underset{\sim}{m} - \underset{\sim}{c} - \underset{\sim}{b}$	
(b)	$\underset{\sim}{m} - \frac{\underset{\sim}{b}}{2}$		(g)	$\frac{\underset{\sim}{c} - \underset{\sim}{b}}{2}$		(l)	$\frac{\underset{\sim}{b} - \underset{\sim}{c}}{2}$	
(c)	$\underset{\sim}{b} - \underset{\sim}{c}$		(h)	$\underset{\sim}{c} - 2\underset{\sim}{m}$		(m)	$\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{m} - \underset{\sim}{e}$	
(d)	$\frac{\underset{\sim}{b}}{2}$		(i)	$\underset{\sim}{b} + \underset{\sim}{c} - \frac{\underset{\sim}{b}}{2}$		(n)	$\frac{\underset{\sim}{b} + \underset{\sim}{c}}{2}$	
(e)	$\underset{\sim}{c} + \underset{\sim}{e} + \underset{\sim}{m}$		(j)	$\underset{\sim}{b} - 2\underset{\sim}{m}$		(o)	$\underset{\sim}{c} + \underset{\sim}{b} - \underset{\sim}{c}$	

CHAPTER 9

Solution of Triangles

What will be learnt?

- Sine Rule
- Cosine Rule
- Area of a Triangle
- Application of Sine Rule, Cosine Rule and Area of a Triangle

List of Learning Standards

bit.ly/2pfbIgk

KEYWORDS

Acute angle	*Sudut tirus*
Obtuse angle	*Sudut cakah*
Sine rule	*Petua sinus*
Cosine rule	*Petua kosinus*
Ambiguous case	*Kes berambiguiti*
Included angle	*Sudut kandung*
Non-included angle	*Sudut bukan kandung*
Three dimension	*Tiga matra*

The architecture in the shape of triangles appears to be very unique in a building. This shape is also used to decorate the walls of building to portray attractive and modern images. We are captivated by the uniqueness of the triangular shapes of this architecture.

However, how can we determine the height of the architecture? What is the information needed to measure the area of each triangle?

Did you Know?

Abu Wafa Muhammad Ibn Muhammad Ibn Yahya Ibn Ismail Buzhani (940-997 M) was a Persian astronomer and mathematician. Abu Wafa learnt trigonometry in Iraq in the year of 959 and developed a few important theories especially in the field of geometry and trigonometry.

For further information:

bit.ly/2q4x80j

SIGNIFICANCE OF THIS CHAPTER

There are various fields which use triangles to solve problems. For example:

- Astronomy field uses the concept of triangle to measure the distance between stars.
- Geography field uses the solution of triangles to measure distance beween various places.
- Satellite field uses triangles in its navigation system.

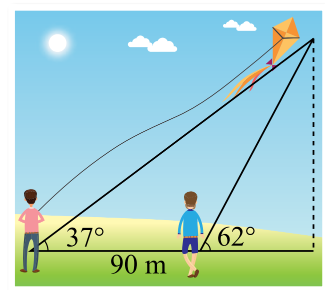
Scan the QR code to watch the video in Masbro village, Melaka.

bit.ly/2VjgB3F

9.1 Sine Rule

Making and verifying conjectures on the relationship between the ratio of length of sides of a triangle with the sine of the opposite angles

In our daily lives, we often face situations involving triangles. For example, the solution to find the height of a kite. When it involves a non-right-angled triangle, Pythagoras Theorem is unsuitable to be used. There are other methods to find the solution of non-right-angled triangles. Let's explore.

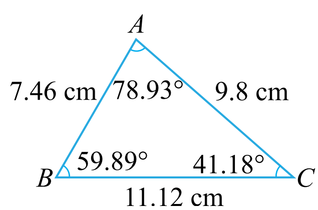
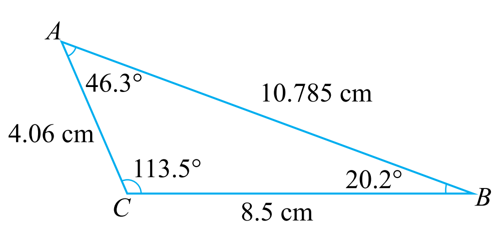


INQUIRY 1 In pairs

Aim: To make conjectures on the relationship between the ratios of length of sides of a triangle with the sine of the opposite angles.

Instructions:

1. Copy or print the table below.
2. Complete the following table based on the given triangles.

	Triangle	$\frac{a}{\sin A}$	$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
(a)	Acute-angled triangle  			
(b)	Obtuse-angled triangle  			

3. Discuss in pairs and state the conjecture on the relationship between the ratio of length of sides of a triangle with the sine of the opposite angles.

From the results of Inquiry 1, it is found that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Is this conjecture valid for all types of acute-angled triangles and obtuse-angled triangles? Let's explore.

Diagram (a) and Diagram (b) is an acute-angled triangle and an obtuse-angled triangle respectively. CD is perpendicular to AB and it is represented by h.

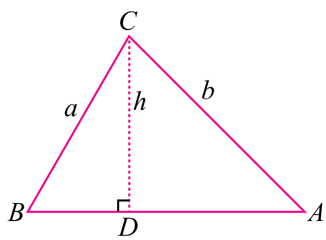


Diagram (a) Acute-angled triangle

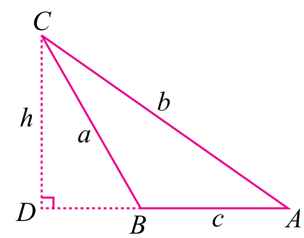


Diagram (b) Obtuse-angled triangle

Consider triangle BCD,

$$\frac{h}{a} = \sin B$$

Then, $h = a \sin B \quad \cdots ①$

Consider triangle ACD,

$$\frac{h}{b} = \sin A$$

Then, $h = b \sin A \quad \cdots ②$

① = ②, $a \sin B = b \sin A$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b}$$

It is observed that for any acute-angled triangle and obtuse-angled triangle, the ratio of length of sides with the sine of the opposite angles are the same. This relationship is known as the **sine rule**.

QR

The relationship between the ratio of length of sides of a triangle with the sine of the opposite angles by using *GeoGebra* software.

bit.ly/2p1IUb2

Sine rule

For any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

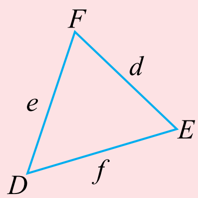
Mind Challenge

What do you get if the sine rule is used in right-angled triangles?

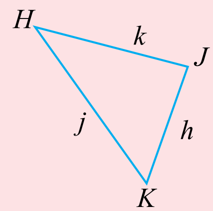
Example 1

Write the sine rule that relates the sides and angles of the following triangles.

(a)



(b)



Solution

(a) $\dfrac{d}{\sin D} = \dfrac{e}{\sin E} = \dfrac{f}{\sin F}$

(b) $\dfrac{h}{\sin H} = \dfrac{j}{\sin J} = \dfrac{k}{\sin K}$

QR

Revision of solution of right-angled triangles.

bit.ly/2B0Crjo

Mind Challenge

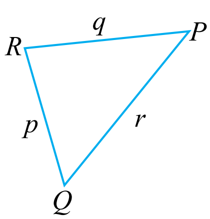
Discuss with your friends and prove that

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

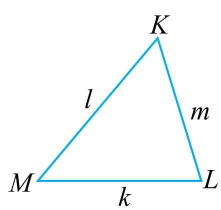
Self Practice 9.1

1. Write the sine rule that relates the sides and angles of the following triangles.

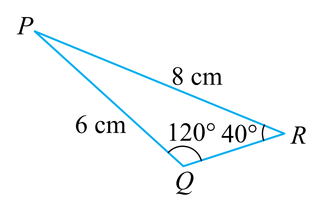
(a)



(b)



(c)



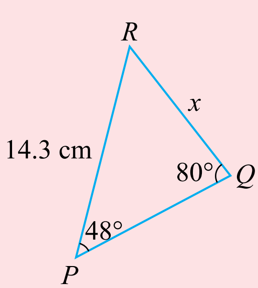
Solving triangles involving sine rule

Solving a triangle means finding the measurements such as length of sides, size of angles, perimeter or area of the triangle. We can solve problems involving triangles by using sine rule.

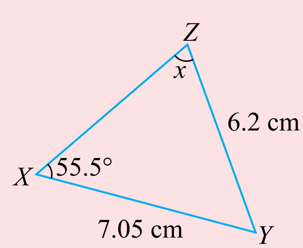
Example 2

Find the value of x in the following triangles.

(a)



(b)



Solution

(a) $\dfrac{x}{\sin 48°} = \dfrac{14.3}{\sin 80°}$

$x = \dfrac{14.3}{\sin 80°} \times \sin 48°$

$= 10.791$ cm

(b) $\dfrac{\sin x}{7.05} = \dfrac{\sin 55.5°}{6.2}$

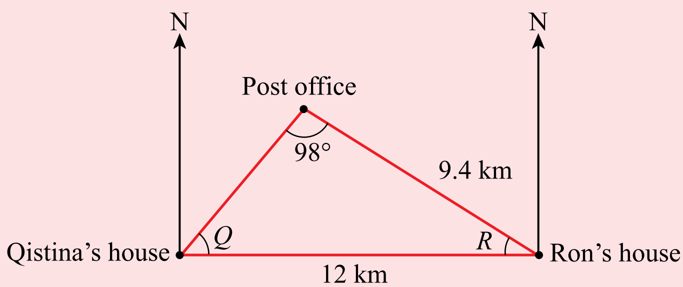
$\sin x = \dfrac{\sin 55.5°}{6.2} \times 7.05$

$= 0.9371$

$x = 69.57°$

Example 3

The diagram below shows the positions of Qistina's house, Ron's house and a post office.



Calculate
(a) the bearing of the post office from Qistina's house,
(b) the bearing of the post office from Ron's house,
(c) the distance from Qistina's house to the post office.

Solution

Assume the positions of the post office, Qistina's house and Ron's house are represented by P, Q and R respectively.

(a) $\dfrac{\sin 98°}{12} = \dfrac{\sin Q}{9.4}$

$\sin Q = \dfrac{\sin 98°}{12} \times 9.4$

$= 0.7757$

$\angle Q = 50.87°$

Bearing P from $Q = 90° - 50.87°$

$= 39.13°$

Thus, the bearing of the post office from Qistina's house is 039.13°.

(b) $\angle R = 180° - \angle P - \angle Q$

$= 180° - 98° - 50.87°$

$= 31.13°$

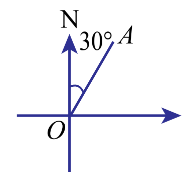
Bearing P from $R = 270° + 31.13°$

$= 301.13°$

Thus, the bearing of the post office from Ron's house is 301.13°.

FLASHBACK

In Geography, bearing is used to indicate the direction of a certain place from a reference point. For example



Bearing A from O in the above diagram is written as 030° or N30°E.

Smart TIPS

In order to solve a triangle by using the sine rule, the following conditions must be learnt:
(a) Two angles and length of one side, or
(b) The lengths of two sides and a non-included angle.

Mind Challenge

What is a non-included angle? Explain.

CHAPTER 9

9.1.2

(c) $\dfrac{r}{\sin 31.13°} = \dfrac{12}{\sin 98°}$ ← *r* represents the distance from Qistina's house to the post office

$r = \dfrac{12}{\sin 98°} \times \sin 31.13°$

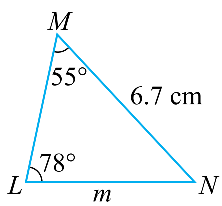
$= 6.265$

Thus, the distance from Qistina's house to the post office is 6.265 km.

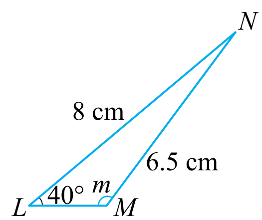
Self Practice 9.2

1. Determine the value of *m* in the following triangles.

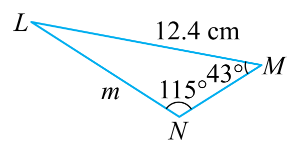
(a)



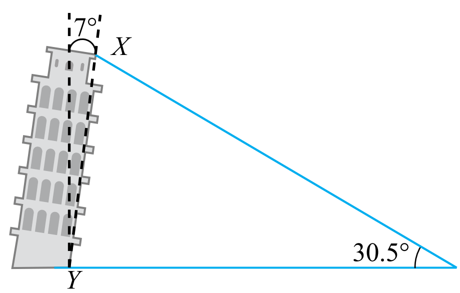
(b)



(c)

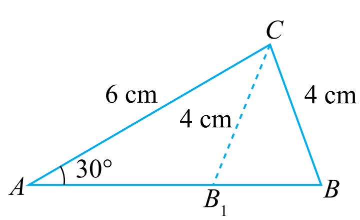


2. The diagram on the right shows a tower which is inclined by 7° from the vertical line. At a distance of 100 m from the side of the tower, the angle of elevation is 30.5°. Estimate the height, *XY*, in m, of the tower.



Determining the existence and solving triangles involving ambiguous cases

The diagram below shows two triangles, *ABC* and AB_1C with the lengths of two sides and a non-included angle given as follow:



MATHEMATICS POCKET

Ambiguity means inexactness or the quality of being open to more than one interpretation.

Based on the diagram above, it is observed that two different triangles can be constructed by using the given non-included angle and lengths of two sides. The two triangles can be constructed by using the same set of information given. This is known as **ambiguous case**.

INQUIRY 2 | In groups | 21st Century Learning

Aim: To determine the conditions for the existence of ambiguous case

Instructions:

1. Scan the QR code or visit the link given.

bit.ly/33c6SiC

2. Given $\angle BAC = 45°$, length of side, $c = 10$ cm and h is the height of the triangle.

3. Drag the slider a to the left and to the right. Observe the changes that take place.

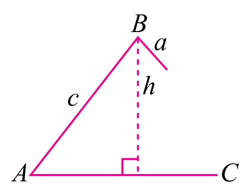
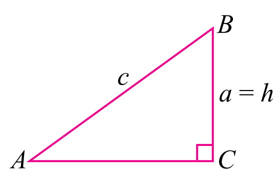
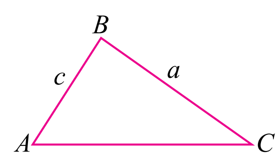
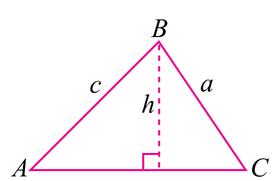
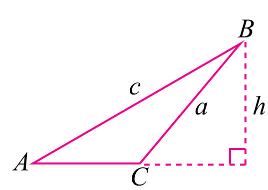
4. Discuss in groups and answer the following questions:
   (a) State your observations when
      (i) $a < h$ (ii) $a = h$ (iii) $a > h$
      (iv) $a < c$ (v) $a = c$ (vi) $a > c$
   (b) Does ambiguous case exist?

5. Each group appoints a representative to do the presentation in the class.

6. Students from other groups are encouraged to ask questions.

7. The teacher will summarise all the presentations.

From the results of Inquiry 2, there are three conditions for the existence of triangles as shown in the following:

No triangle exists		$a < h$
One triangle exists	 $a = h$	 $a \geqslant c$
Two triangles exist	 $h < a < c$	 $h < a < c$

Ambiguous case exists if:

(a) Given the lengths of two sides, a and c, and a non-included angle, $\angle A$ which is acute.

(b) The side which is opposite the non-included angle, a is shorter than the other side, c, but it is longer than the height, h, of the triangle.

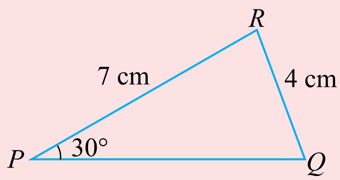
9.1.3 9.1.4

CHAPTER 9

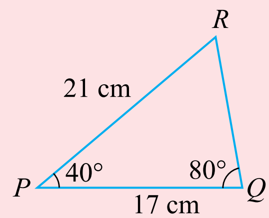
Example 4

Determine whether ambiguous case exists for the following triangles. Explain your answer.

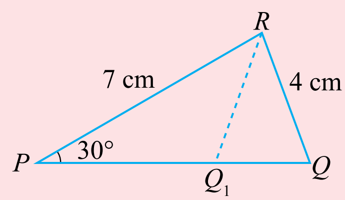
(a)



(b)



Solution



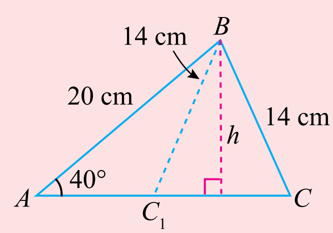
(a) Yes, ambiguous case exists in triangle PQR because the non-included angle $\angle QPR = 30°$ and the side RQ is shorter than the side PR but it is longer than the height of triangle.

(b) Ambiguous case does not exist because the angles of the two sides are given.

Example 5

In the triangle ABC, $\angle BAC = 40°$, $AB = 20$ cm and $BC = 14$ cm. Calculate the possible values of $\angle C$ and $\angle B$.

Solution



Determine whether ambiguous case exists for triangle ABC.

Height, $h = 20 \sin 40°$

$= 12.856$ cm

Since $h < BC < AB$, then ambiguous case exists.

Look at the sketch of the triangle ABC in the diagram on the right. The two triangles that exist are ABC and ABC_1.

For triangle ABC,

$$\frac{\sin \angle C}{20} = \frac{\sin 40°}{14}$$

$$\sin \angle C = \frac{20 \sin 40°}{14}$$

$$= 0.9183$$

$\angle C = 66.68°$ $\qquad$ $\angle C_1 = 180° - 66.68°$

$= 113.32°$

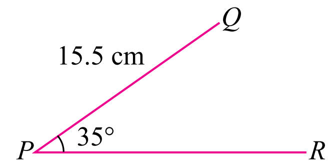
$\angle B = 180° - 40° - 66.68°$ $\qquad$ $\angle B_1 = 180° - 40° - 113.32°$

$= 73.32°$ $\qquad$ $= 26.68°$

Self Practice 9.3

1. For each of the following triangles, determine whether ambiguous case exists.
   (a) ΔABC; $\angle B = 62.5°$, $BC = 14.5$ cm and $AC = 10$ cm.
   (b) ΔPQR; $\angle R = 28°$, $QR = 8.2$ cm and $PQ = 11.4$ cm.

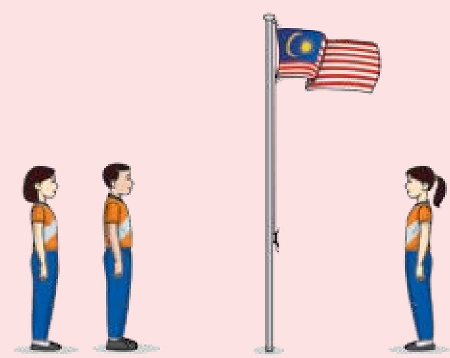
2. The diagram on the right shows an incomplete triangle PQR. $PQ = 15.5$ cm and $\angle QPR = 35°$. Given $QR = 10.5$ cm,
   (a) find the possible values of $\angle QRP$,
   (b) hence, find the possible lengths of PR.



Solving problems related to triangles using the sine rule

Example 6

Azyan and Christine stand straight in front of a flag post as shown in the diagram. The elevation angle of top of the pole from Azyan is 36° whereas the elevation angle of top of the pole from Christine is 50°. Badrul is standing on the left side of the flag pole and the elevation angle of the top of the pole from him is the same as Christine. The distance between Azyan and Christine is 35 m. Find the distance between Azyan and Badrul if the height of three of them are the same.

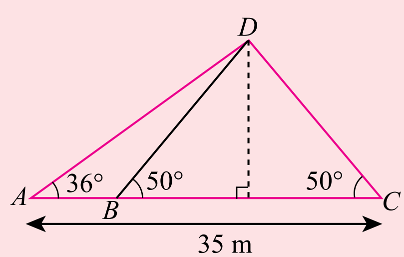


Solution

Represent the positions of Azyan, Badrul, Christine and the top of the pole with A, B, C and D respectively.

$\angle ADC = 180° - 50° - 36°$
$= 94°$

$\angle BDC = 180° - 50° - 50°$
$= 80°$



$$\frac{DC}{\sin 36°} = \frac{35}{\sin 94°}$$

$$DC = \frac{35}{\sin 94°} \times \sin 36°$$
$$= 20.6227 \text{ m}$$

$$\frac{BC}{\sin 80°} = \frac{20.6227}{\sin 50°}$$

$$BC = \frac{20.6227}{\sin 50°} \times \sin 80°$$
$$= 26.5120 \text{ m}$$

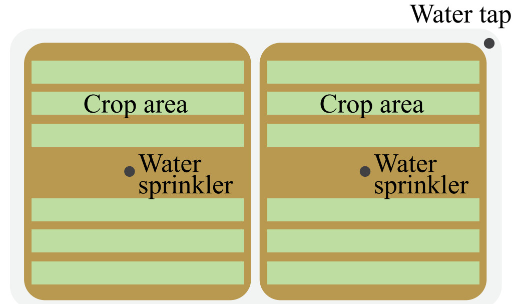
$AB = 35 \text{ m} - 26.5120 \text{ m}$
$= 8.488 \text{ m}$

Then, the distance between Azyan and Badrul is 8.488 m.

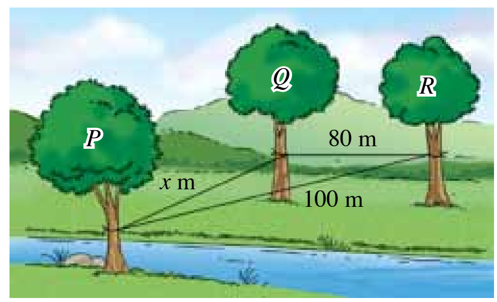
9.1.3 9.1.4 9.1.5

Self Practice 9.4

1. Encik Samad makes a plan for his vegetable farm as shown in the diagram. Encik Samad wants to install two water sprinklers in the middle of the farm. The water tap which controls the water sprinklers is installed at one corner of the farm. The distance between the two water sprinklers is 6 m and the distance between the water tap and the nearest water sprinkler is 5 m. The angle formed between the tap and both water sprinklers is 25°. Calculate the distance between the tap and the furthest water sprinkler.



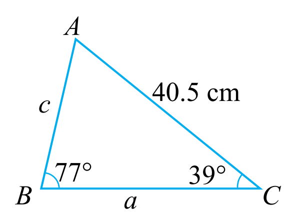
2. A group of scouts organised an activity on crossing a river during a motivational camp. They tied a rope from tree P to tree Q and tree R on the other side of the river as shown in the diagram. The distance between tree Q and tree R is 80 m and a 50° angle is formed between tree Q and tree R at P. Find the value of x, the distance from tree P to tree Q.



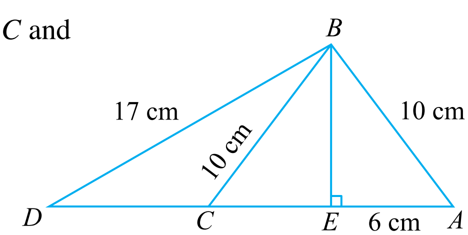
Intensive Practice 9.1

Scan the QR code or visit bit.ly/35oe3pQ for the quiz

1. The diagram on the right shows a triangle ABC such that $\angle B = 77°$, $\angle C = 39°$ and $AC = 40.5$ cm. Calculate the value of $\angle A$, a and c.

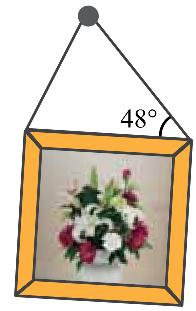


2. The diagram on the right shows a triangle ABD. Point C and point E lie on the straight line AD.
   (a) Find the lengths of BE, CE and DE.
   (b) Calculate $\angle EAB$, $\angle BCE$, $\angle BCD$, $\angle ABD$ and $\angle CBD$.
   (c) Explain the ambiguous case in the diagram on the right.

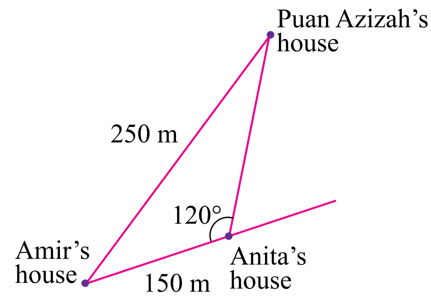


3. In the obtuse-angled triangle PQR, $PR = 14$ cm, $QR = 6\sqrt{3}$ cm and $\angle QPR = 40°$.
   (a) State the obtuse angle and find the size of that angle.
   (b) Calculate the length of PQ.

4. The diagram on the right shows a square picture frame which is hung by Amira using two pieces of ropes. Amira finds that the picture frame inclines towards the right. The angle formed between the longer rope and the frame is 48°. The lengths of the rope are 20 cm and 15 cm respectively. Calculate the perimeter of the frame.

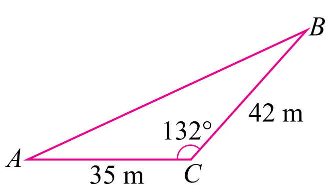


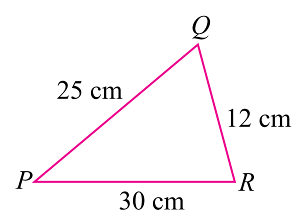
5. The diagram on the right shows the position of Puan Azizah's house and the houses of her two children, Amir and Anita. Another child, Aida wants to build her house such that all the three houses of the siblings are collinear and the distance from her house and Anita's house to Puan Azizah's house is the same. Find the distance between Anita's house and Aida's house.



9.2 Cosine Rule

Observe the diagrams below.





How do you determine the length of *AB* and angle *PQR*? Can both triangles be solved by using the sine rule?

When the lengths of two sides and an included angle or the lengths of three sides are given, the triangle cannot be solved by using the sine rule. The triangles with such conditions can be solved by using the cosine rule.

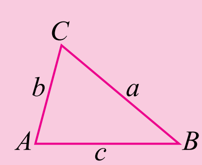
Cosine rule

For any triangle *ABC*,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$



CHAPTER 9

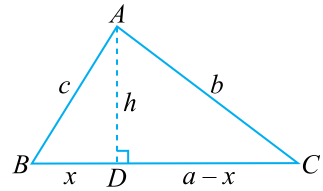
9.2.1

Verifying the cosine rule

Is the cosine rule true for all types of triangles? Let's explore.

Consider the triangle ABC in the diagram. By using the Pythagoras Theorem in the triangle ACD,

$b^2 = h^2 + (a - x)^2$
$b^2 = h^2 + a^2 - 2ax + x^2 \quad \cdots ①$



Use the Pythagoras Theorem in the triangle ABD,

$c^2 = h^2 + x^2$
$h^2 = c^2 - x^2 \quad \cdots ②$

Substitute ② into ①.

$b^2 = c^2 - x^2 + a^2 - 2ax + x^2$
$b^2 = c^2 + a^2 - 2ax \quad \cdots ③$

In the triangle ABD,

$\cos B = \frac{x}{c}$
$x = c \cos B$

Substitute $x = c \cos B$ into ③.

$b^2 = c^2 + a^2 - 2ac \cos B$

Mind Challenge
Can the cosine rule be used on the right-angled triangles? Explain.

This equation is one of the formulae of cosine rule. Try to verify the cosine rule for obtuse-angled triangle.

Solving triangles involving the cosine rule

Cosine rule can be used to find the length or unknown angle in a triangle when the lengths of two sides and an included angle or the lengths of three sides are given.

Example 7

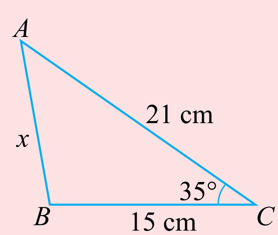
In the triangle ABC, $AC = 21$ cm, $BC = 15$ cm and $\angle C = 35°$. Find the length of AB.

Solution

Sketch the triangle ABC.
By using the cosine rule,

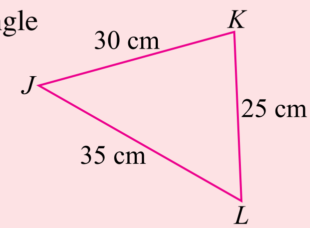
$x^2 = 15^2 + 21^2 - 2(15)(21) \text{ kos } 35°$
$= 225 + 441 - 630 \text{ kos } 35°$
$= 149.9342$

Then, $x = \sqrt{149.9342}$
$= 12.245$ cm



Example 8

The diagram on the right shows a triangle *JKL* with the length of *JK* = 30 cm, *KL* = 25 cm and *JL* = 35 cm. Find the value of $\angle KJL$.



Solution

By using the cosine rule,

$25^2 = 30^2 + 35^2 - 2(30)(35) \cos \angle KJL$

$\cos \angle KJL = \dfrac{30^2 + 35^2 - 25^2}{2(30)(35)}$

$= 0.7143$

$\angle KJL = 44.41°$

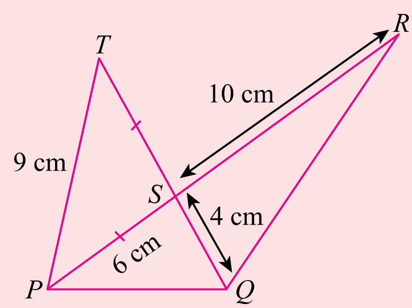
Smart TIPS

To find the angles, the formulae of cosine rule can be written as follow:

- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
- $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example 9

In the diagram on the right, *QST* and *PSR* are straight lines. Find the length of *QR*.



Solution

By using the cosine rule,

$9^2 = 6^2 + 6^2 - 2(6)(6) \cos \angle PST$

$\cos \angle PST = \dfrac{6^2 + 6^2 - 9^2}{2(6)(6)}$

$= -0.1250$

$\angle PST = 97.18°$

$QR^2 = 4^2 + 10^2 - 2(4)(10) \cos 97.18°$

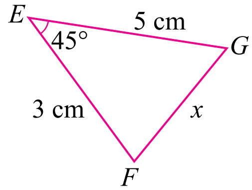
$= 125.999$

$QR = 11.225$ cm

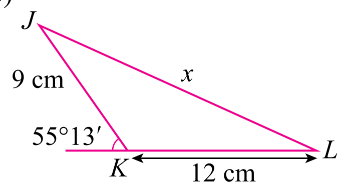
Self Practice 9.5

1. Find the value of x in the following triangles.

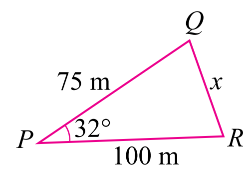
(a)



(b)



(c)

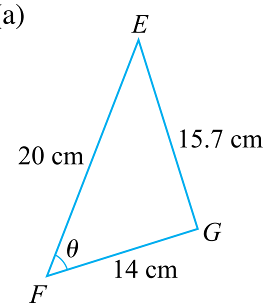


CHAPTER 9

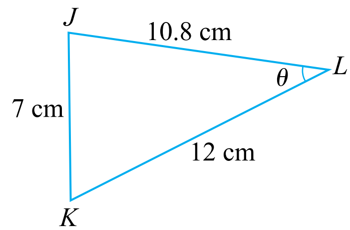
9.2.2

2. Find the value of θ in the following triangles.

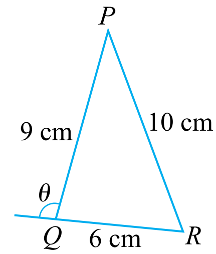
(a)



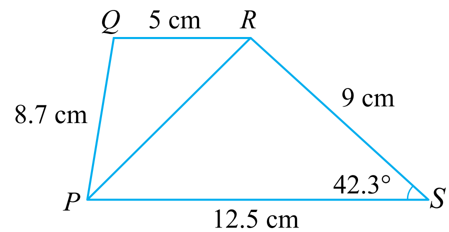
(b)



(c)



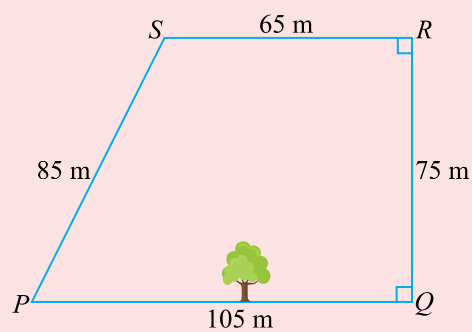
3. The diagram on the right shows a quadrilateral *PQRS*. Find the angle *PQR*.



Solving problems involving the cosine rule

Example 10

Mr. Sivaraja has a plot of land in the shape of trapezium *PQRS* as shown in the diagram on the right. He puts up a fence around the land.There is a tree at a distance of 50 m from the vertex *Q*. Mr. Sivaraja wants to divide the land into two parts by putting some additional fence from the vertex *S* to the tree. Calculate the length of fence put up by Mr. Sivaraja.



Solution

$SQ = \sqrt{65^2 + 75^2}$

$= 99.2472$ m

$99.2472^2 = 85^2 + 105^2 - 2(85)(105) \cos \angle SPQ$

$\cos \angle SPQ = \dfrac{85^2 + 105^2 - 99.2472^2}{2(85)(105)}$

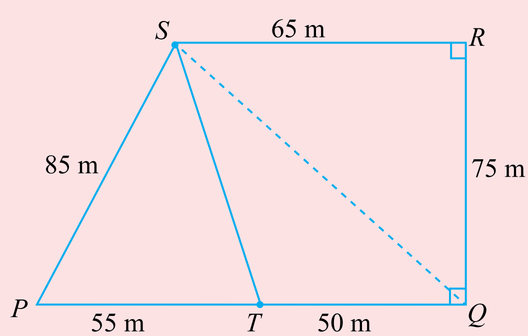
$\angle SPQ = 61.93°$

$ST^2 = 55^2 + 85^2 - 2(55)(85) \cos 61.93°$

$= 5850.3581$

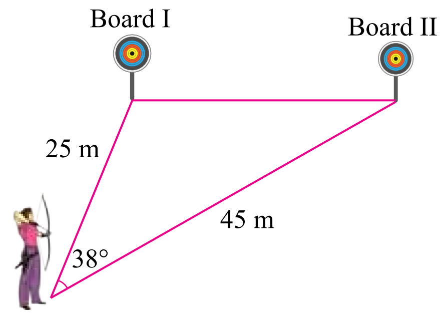
$ST = 76.488$ m

The length of the additional fence is 76.488 m.

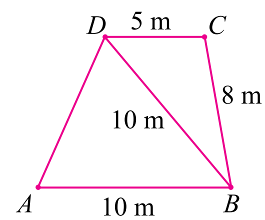


Self Practice 9.6

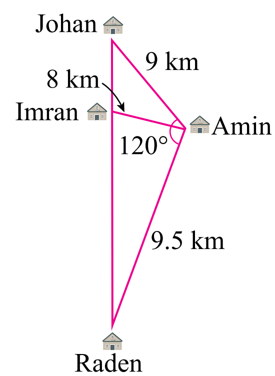
1. Farid carried out archery training on a field. The diagram on the right shows two target boards which have to be struck by arrows. The distance between Farid and board I and board II are 25 m and 45 m respectively. The standing position of the shooting is 38° between board I and board II. Calculate the distance between board I and board II.



2. Frank planted four iron rods in the ground and installed wires to build clotheslines. The sketch of the clotheslines built by Frank is shown in the diagram on the right. The wire *AB* is parallel to the wire *DC*. Calculate the total length of the wire used by Frank.



3. The diagram on the right shows the positions of the house of four friends, Amin, Imran, Johan and Raden. During Hari Raya, Amin wants to visit all the three houses of his friends. Amin intends to pick up Imran and then send him back home before he returns to his house. What is the total distance travelled by Amin for the whole journey?

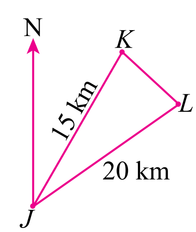


Intensive Practice 9.2

Scan the QR code or visit bit.ly/2VvtloA for the quiz

1. A card is in the shape of a parallelogram. Given the lengths of the diagonals of the card are 6 cm and 10 cm respectively. The acute angle between the diagonals are 62°. Calculate the lengths of sides of the card.

2. The diagram on the right shows the positions of three towns, *J*, *K* and *L*. Given the bearing of *K* from *J* is 020° and bearing of *L* from *J* is 055°, find the distance between town *K* and town *L*.

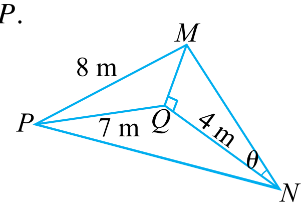


9.2.3

CHAPTER 9

3. Bunga Raya ship left a port and sailed east for a distance of 28 km. Bunga Orkid ship left the same port and sailed for 49 km. If the final distance between the two ships is 36 km, find the angle between the routes of Bunga Raya ship and Bunga Orkid ship.

4. The diagram on the right shows a pond in the shape of triangle MNP. Given $\cos \theta = \frac{4}{5}$, $MP = 8$ m, $PQ = 7$ m and $QN = 4$ m. Encik Raja decorates the pond by arranging stones around the pond. Calculate the length of stones arranged by Encik Raja around the pond.

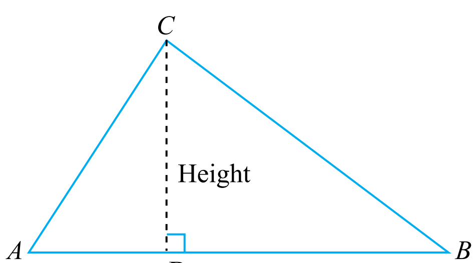


9.3 Area of a Triangle

Deriving the formula and determining the area of a triangle

The diagram on the right shows the shape of a window of a building and it is in the shape of a triangle. What is the information required to calculate the area of the window in the diagram and what is the formula that you will use to determine the area of the window?

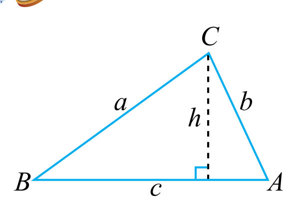
You have learnt that the area of the triangle can be determined by using the following formula:



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

The formula for the area of triangle can be used when the length of base and height of triangle are given. How do you find the area of a triangle without knowing the length of the base and the height? Let's explore the method used to derive the formula for the area of a triangle.

FLASHBACK



$h = a \sin B$
$h = b \sin A$

9.3.1

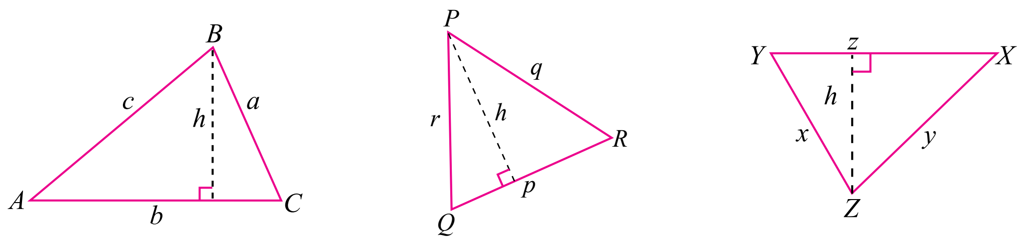
CHAPTER 9

INQUIRY 3 In groups 21st Century Learning

Aim: To derive the formula for the area of a triangle

Instructions:

1. Start this activity in pairs.
2. Observe the following shapes of triangles.



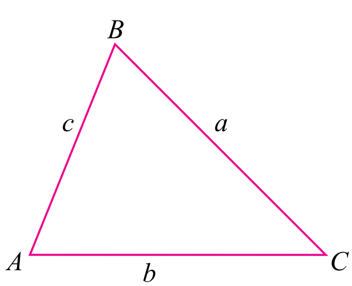
Triangle I　　Triangle II　　Triangle III

3. Find the height of each of the triangles by using trigonometric ratios.
4. Then, copy and complete the following table based on the triangles above.

Triangle	Base	Height	Area
I	AC		
II			
III			

5. Compare the formulae for the area of the three triangles and state the conclusion based on your findings.
6. Form a few groups. Then, each pair will share the results and conclusions in their respective group.

From the results of Inquiry 3, if the lengths of two sides and an included angle are only given, the area of a triangle can be calculated by using the following formulae:



$$\text{Area} = \frac{1}{2}ab \sin C$$
$$= \frac{1}{2}ac \sin B$$
$$= \frac{1}{2}bc \sin A$$

QR

Methods to derive the formulae for the area of a triangle.

bit.ly/316kXNk

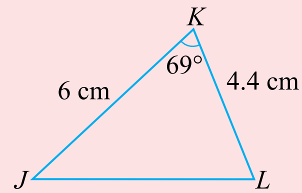
CHAPTER 9

9.3.1

Example 11

Find the area of triangle *JKL* in the diagram on the right.

Solution



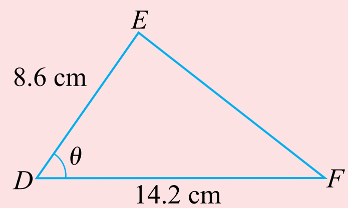
Included angle = 69°

$$\text{Area} = \frac{1}{2}(6)(4.4)\sin 69°$$
$$= 12.323 \text{ cm}^2$$

Example 12

The area of triangle *DEF* is 50 cm^2. Given $DE = 8.6$ cm, $DF = 14.2$ cm and $\angle EDF = \theta$, find the value of θ.

Solution

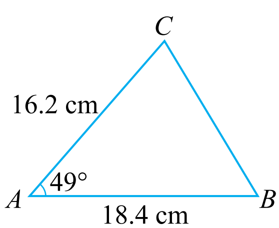


$$\frac{1}{2}(8.6)(14.2)\sin\theta = 50$$
$$\sin\theta = \frac{50}{61.06}$$
$$\theta = 54.97°$$

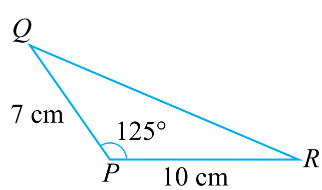
Self Practice 9.7

1. Find the area of the following triangles.

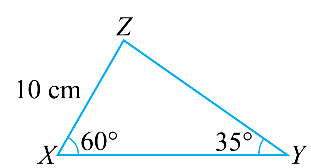
(a)



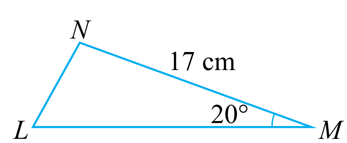
(b)



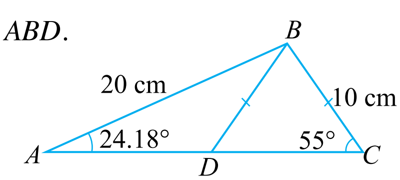
(c)



2. In the diagram on the right, the area of triangle *LMN* is 78.72 cm^2. Find the length of *LM*.



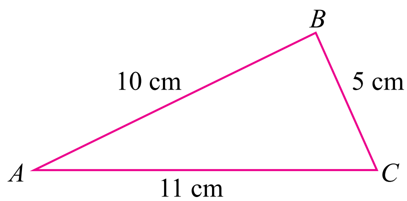
3. The diagram on the right shows triangle *BCD* and triangle *ABD*. Find the area of triangle *ABD*.



4. Find the area of triangle *XYZ*, given $x = 5.5$ m, $z = 7$ m and $\angle Y = 70°30'$.

Determining the area of a triangle using the Heron's formula

Consider the following triangle *ABC*:



When only the lengths of each side are given, the area of the triangle can be determined by using the Heron's formula.

The solving steps are as follows:

Step 1

Calculate the semi perimeter, $s = \dfrac{a + b + c}{2}$, such that a, b and c are the lengths of sides.

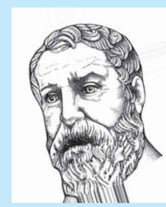
Step 2

Substitute the values of s, a, b and c into the following formula:

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}$$

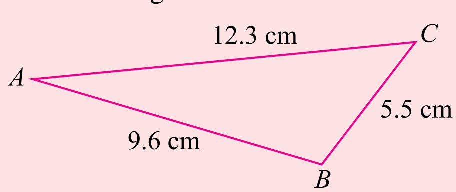
Mathematics Museum

Hero of Alexandria also known as Heron is a Greek mathematician. Heron's formula was named after him and had been written in his book entitled "Metrica".



Example 13

Find the area of the triangle below.



Solution

$$s = \frac{5.5 + 9.6 + 12.3}{2}$$
$$= 13.7$$
$$\text{Area} = \sqrt{13.7(13.7 - 5.5)(13.7 - 9.6)(13.7 - 12.3)}$$
$$= 25.39 \text{ cm}^2$$

QR

Verification of Heron's formula.

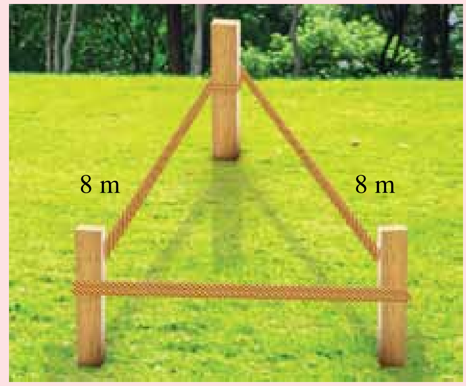
bit.ly/2WrkvbM

CHAPTER 9

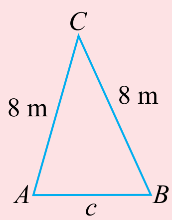
9.3.2

Example 14

A group of scouts planted three pieces of wood in a camping ground to build a fire pit. A rope of length 22 m is used to tie around those woods as shown in the diagram. The rope formed an isosceles triangle. The length of the rope at the side of equal length is 8 m. Calculate the area of the region for them to build the fire pit.



Solution



Given the perimeter of triangle = 22 m, $a = 8$ m, $b = 8$ m.

$c = 22 - 8 - 8$
$= 6$ m

$s = \frac{22}{2}$
$= 11$

Area $= \sqrt{11(11-8)(11-8)(11-6)}$
$= 22.249 \text{ m}^2$

Thus, the area of the region for building the fire pit is 22.249 m^2.

Alternative Method

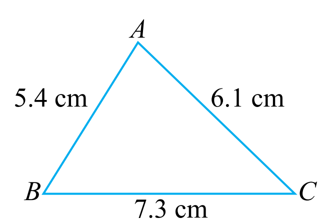
$6^2 = 8^2 + 8^2 - 2(8)(8)\cos\angle ACB$

$\angle ACB = 44.05°$

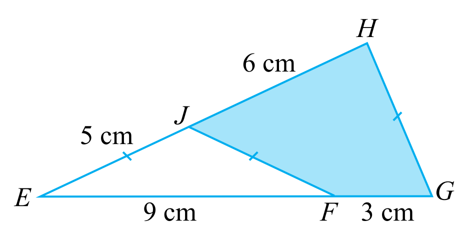
Area $= \frac{1}{2}(8)(8)\sin 44.05°$
$= 22.249 \text{ m}^2$

Self Practice 9.8

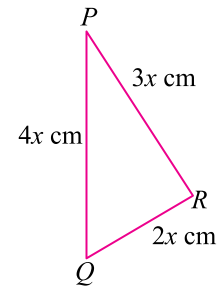
1. The diagram on the right shows a triangle *ABC* such that $AB = 5.4$ cm, $AC = 6.1$ cm and $BC = 7.3$ cm. Calculate the area, in cm^2, of the triangle *ABC*.



2. The diagram on the right shows two triangles, *EFJ* and *EGH*. *EFG* and *EJH* are straight line. Calculate the area, in cm^2, of the shaded region.



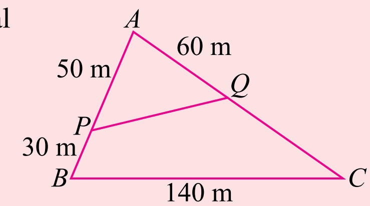
3. Mr. Sammy wants to paint the wall of his room. He draws a triangular shape on the wall and he will paint the triangle with green paint. The shape of the triangle is as shown in the diagram on the right. The lengths of sides of the triangle are $2x$ cm, $3x$ cm and $4x$ cm respectively. The area is $\sqrt{135}$ cm^2. Find the value of x.



Solving problems involving areas of triangles

Example 15

The diagram on the right shows the plan of a plot of agricultural land in the shape of triangle ABC owned by Encik Munzir. The part APQ will be planted with chillies and the remaining part will be planted with cabbage. Given $AP = 50$ m, $AQ = 60$ m, $AB = 80$ m, $AC = 130$ m and $BC = 140$ m, find the area of land which will be planted with cabbage.



Solution

Assume L_1 as the area of triangle ABC and L_2 as the area of triangle APQ.

Heron's formula to find L_1.

$$s = \frac{80 + 130 + 140}{2}$$
$$= 175$$
$$L_1 = \sqrt{175(175 - 80)(175 - 130)(175 - 140)}$$
$$= 5117.0670 \text{ m}^2$$

Use the formula, area $= \frac{1}{2}bc \sin A$ to find $\angle BAC$.

$$\frac{1}{2}(80)(130) \sin \angle BAC = 5117.0670$$
$$\sin \angle BAC = \frac{5117.0670}{\frac{1}{2}(80)(130)}$$
$$\angle BAC = 79.75°$$

Alternative Method

$$\cos A = \frac{80^2 + 130^2 - 140^2}{2(80)(130)}$$
$$A = 79.75°$$

Use the formula $= \frac{1}{2}pq \sin A$ to find L_2.

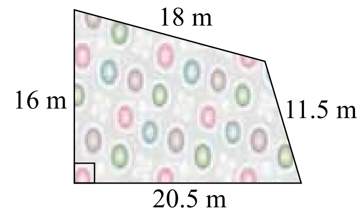
$$L_2 = \frac{1}{2}(50)(60) \sin 79.75°$$
$$= 1476.0610 \text{ m}^2$$

Thus, the area of land which will be planted with cabbage $= L_1 - L_2$
$= 5117.0670 - 1476.0610$
$= 3641.006$ m^2

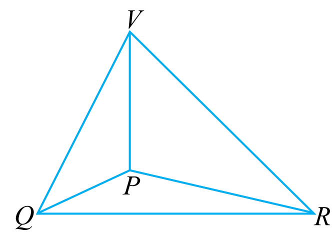
9.3.2 9.3.3

Self Practice 9.9

1. Mr. Khan won a tender to install carpet in an office. Calculate the area of carpet required to fill up the office space as shown in the diagram on the right.



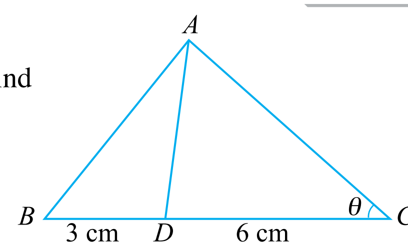
2. The diagram on the right shows a decoration in the shape of pyramid. The decoration has a triangular base PQR. Vertex V is vertically above vertex P. Given $PQ = 4$ cm, $PV = 10$ cm, $VR = 15$ cm and $\angle VQR = 80°$, calculate the area of the inclined surface of the decoration.



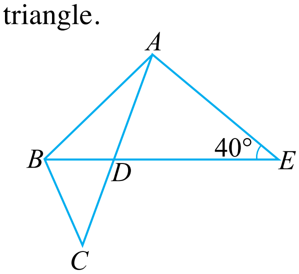
Intensive Practice 9.3

Scan QR code or visit bit.ly/2ohScA1 for the quiz

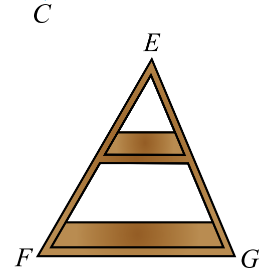
1. The diagram on the right shows a triangle ABC.
Given the area of triangle $ABC = 18$ cm^2 and $\sin \theta = \frac{2}{3}$, find
(a) the length of AC,
(b) the area of triangle ABD.



2. A regular pentagon has sides of 5 cm each. Find the area of the regular pentagon.
3. Mei Ling wants to prepare a greeting card in the shape of a triangle. The area of the card is 30 cm^2 and the lengths of two sides are 8 cm and 11 cm. Find the possible length of the third side.
4. The length of the sides of a triangle are $3x$ cm, $(x – 1)$ cm and $(3x + 1)$ cm. Given the perimeter of the triangle is 63 cm. Calculate the area, in cm^2, of the triangle.
5. Pooja fenced up a plot of land in the shape as shown in the diagram on the right. Given $BD = 5$ m, $BC = 7$ m, $CD = 8$ m and $AE = 12$ m. BDE and ADC are straight lines. Calculate the area of land fenced up by Pooja.



6. The diagram on the right shows a decoration rack in the shape of a triangle, EFG. Given $FG = 15$ cm, $EG = 16$ cm and $EF = 17$ cm, find the height of the rack.



CHAPTER 9

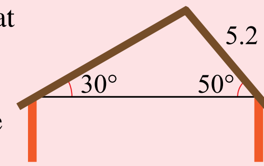
9.3.3

9.4 Application of Sine Rule, Cosine Rule and Area of a Triangle

Solving problems involving triangles

Example 16 MATHEMATICS APPLICATION

Mr. Tan wants to paint the roof of the garage. The diagram on the right is a sketch of the front view of the roof of the garage. He found out that the wood on one part of the roof is longer than the wood on the other part of the roof.



(a) Calculate the length of wood on the longer part of the roof and the distance between both the walls of the garage.

(b) What is the area of the front roof in the shape of triangle, in m^2, which will be painted by Mr. Tan?

Solution

1. Understanding the problem

- Length of one side of the roof = 5.2 m.
- Two angles given 30° and 50°.
- Calculate the length of two sides and the area of triangle.

2. Planning a strategy

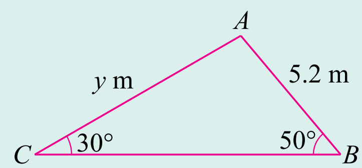
- Draw the triangle ABC which represents the front view of the roof of the garage.
- Length of one side of the roof, $AC = y$ is calculated using sine rule.
- Determine $\angle BAC$ and hence calculate BC using cosine rule.
- Find the area of triangle ABC using the formula:

$$\text{Area} = \frac{1}{2}ab \sin C$$

or Heron's formula.

3. Implementing the strategy

(a)



By using sine rule,

$$\frac{y}{\sin 50°} = \frac{5.2}{\sin 30°}$$

$$y = \frac{5.2}{\sin 30°} \times \sin 50°$$

$$= 7.967 \text{ m}$$

Thus, the length of the other part of the roof is 7.967 m.

$$\angle BAC = 180° - 30° - 50°$$
$$= 100°$$

By using cosine rule,

$$BC^2 = 5.2^2 + 7.967^2 - 2(5.2)(7.967) \cos 100°$$

$$BC = 10.24 \text{ m}$$

Thus, the distance between both walls of the garage is 10.24 m.

(b) Area of triangle ABC

$$= \frac{1}{2}(5.2)(10.24) \sin 50°$$

$$= 20.40 \text{ m}^2$$

Thus, the triangular area which will be painted by Mr. Tan is 20.40 m^2.

CHAPTER 9

9.4.1

4. Making a conclusion

Using Heron's formula,

$s = \frac{5.2 + 7.967 + 10.24}{2} = 11.7035$ m

Area

$= \sqrt{11.7035(11.7035 - 5.2)(11.7035 - 7.967)(11.7035 - 10.24)}$

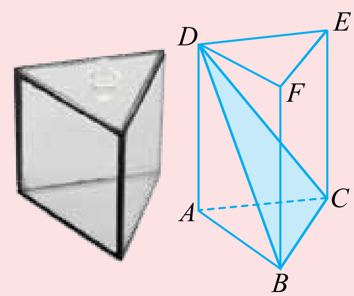
≈ 20.40 m^2

The value of *AC*, *BC* and the area calculated are valid.

Example 17

The diagram on the right shows a glass prism and the sketch of the prism. The cross section of the prism is an equilateral triangle sides of 6 cm and the height of the prism is 8 cm. Calculate

(a) the angle between *BD* and *CD*,
(b) the area of *BCD*,
(c) the angle between the plane *BCD* and the vertical plane *BCEF*.



Solution

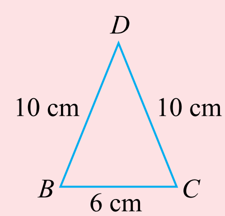
(a) $CD = \sqrt{6^2 + 8^2}$
$= 10$ cm

$6^2 = 10^2 + 10^2 - 2(10)(10) \cos \angle BDC$

$\cos \angle BDC = \frac{10^2 + 10^2 - 6^2}{2(10)(10)}$

$\angle BDC = 34.92°$

Thus, the angle between *BD* and *CD* is 34.92°.



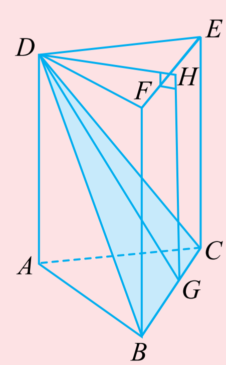
(b) Area of triangle $BCD = \frac{1}{2}(10)(10) \sin 34.92°$
$= 28.622$ cm^2

(c) Based on the diagram on the right, the angle between the plane *BCD* and vertical plane *BCEF* is $\angle DGH$.

$DH = \sqrt{6^2 - 3^2}$
$= 5.1962$

$\tan \angle DGH = \frac{5.1962}{8}$

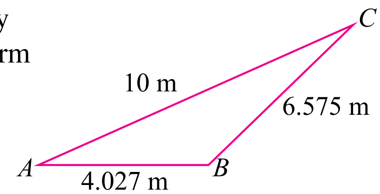
$\angle DGH = 33°$



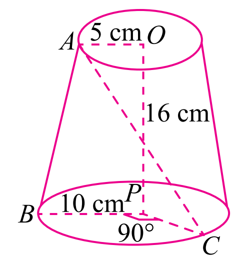
CHAPTER 9

Self Practice 9.10

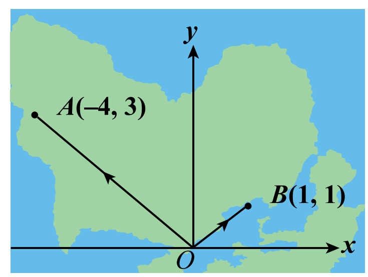
1. In an examination hall, the tables of Daniel, Darvin and Cindy are at positions A, B and C respectively. These three points form a triangle as shown in the diagram on the right. The distance between Daniel's table and Cindy's table is 10 m, Daniel's table and Darvin's table is 4.027 m whereas Darvin's table and Cindy's table is 6.575 m. Prove that the sum of the interior angles of the triangle formed is 180°.



2. The diagram on the right shows a children's toy in the shape of a cone and its upper portion is cut off. The surfaces in the shape of circles, centre O and centre P are horizontal and the OP axis is vertical. There is a straight line that joins A to C. Given $OA = 5$ cm, $PB = 10$ cm, $OP = 16$ cm and $\angle BPC = 90°$, calculate
   (a) the length of AC,
   (b) the area of plane ABC.



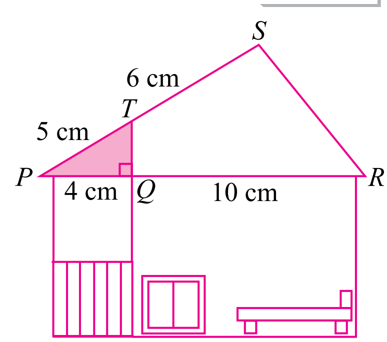
3. The positions of two towns, A and B, are shown in the Cartesian plane in the diagram on the right. Find the angle between the position vector of town A and town B relative to the origin O. Hence, find the area of the region in the shape of the triangle OAB.



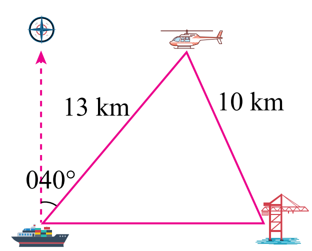
Intensive Practice 9.4

Scan QR code or visit bit.ly/2IG2l0m for the quiz

1. The diagram on the right shows the front view of a doll house built by Melly. The coloured part is the veranda roof of the doll house. PTS and PQR are straight lines.
   (a) Calculate the area of the roof $QRST$.
   (b) Point U lies on PR such that $SU = SR$, calculate $\angle SUP$.



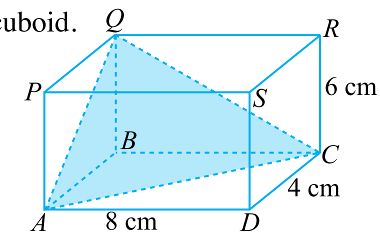
2. The diagram on the right shows the position of an oil rig, a tanker and a helicopter. The bearing of the helicopter from the tanker is 40°. Given the distance between the helicopter and the tanker is 13 km whereas the distance between the helicopter and the oil rig is 10 km. Calculate the distance, in km, between the tanker and the oil rig.



9.4.1

CHAPTER 9

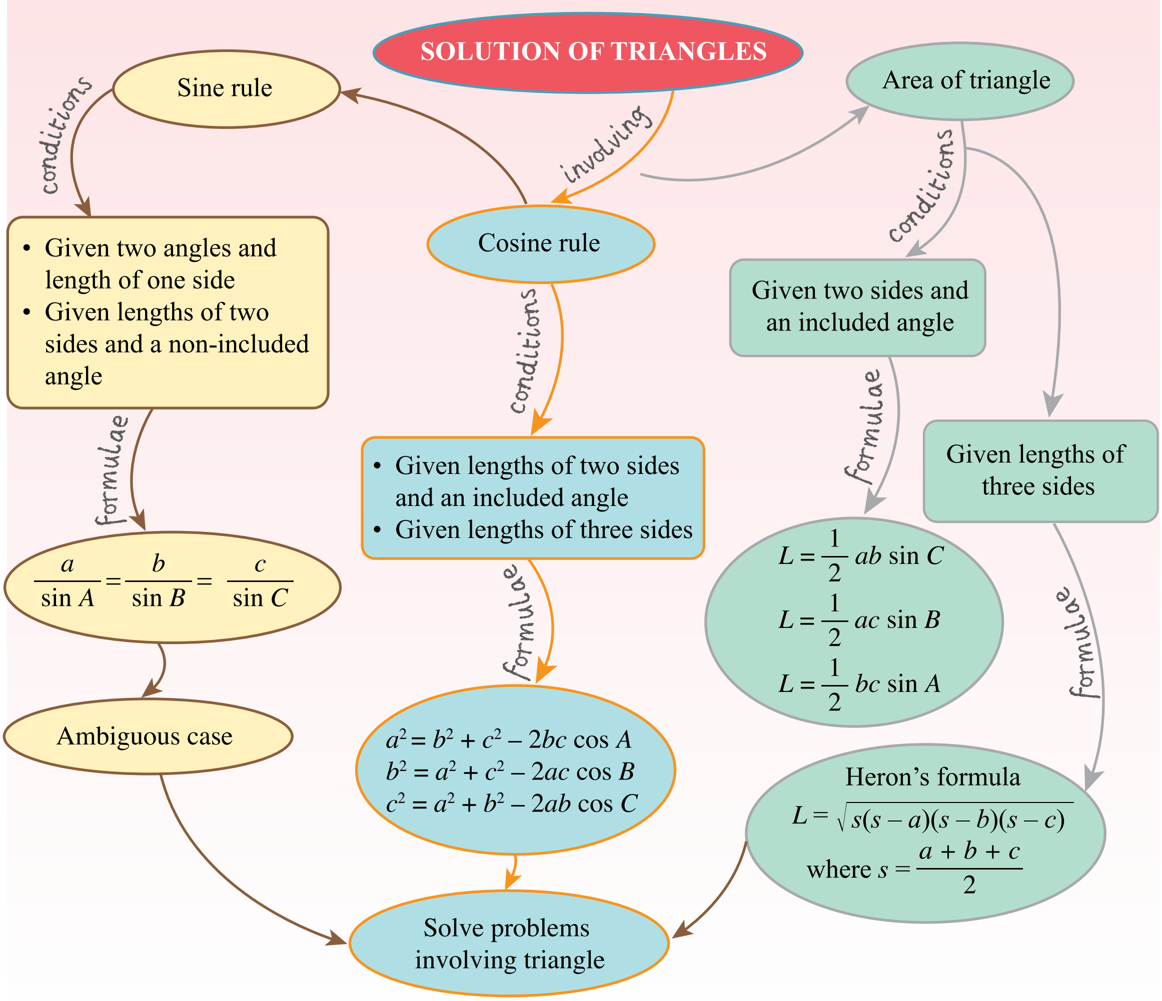
3. The diagram on the right shows a gift box in the shape of a cuboid.
   (a) Calculate the area of the plane ACQ.
   (b) Hence, state another plane which has the same area as the plane ACQ.



4. A ship sailed for 20 km to Port Bentara at the bearing of 120° from Port Astaka. Then, the ship sailed for 30 km to Port Cindai at the bearing of 225° from Port Bentara. Calculate the distance and the bearing of Port Cindai from Port Astaka.

5. The elevation angle of the peak of a mountain from Arman is 20°. Then, Arman walks horizontally towards the mountain that is 800 m away and the elevation angle becomes 45°. Estimate the height of the mountain from the level of Arman's position.

SUMMARY OF CHAPTER 9



CHAPTER 9

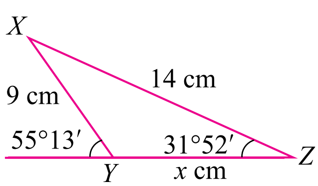
WRITE YOUR JOURNAL

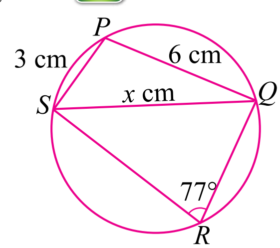
1. Draw a flow chart which shows the steps used in choosing the suitable rules to find
   (a) length of sides or size of angles of a triangle,
   (b) the area of a triangle.

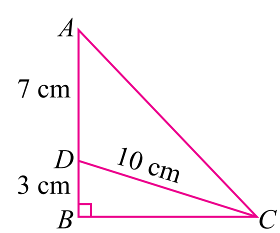
2. Surf the internet to get
   (a) the examples of usage of sine rule, cosine rule and the formula of the area of triangles in our daily lives,
   (b) the area of the Kuala Lumpur Golden Triangle, India Golden Triangle and Bermuda Triangle.

MASTERY PRACTICE

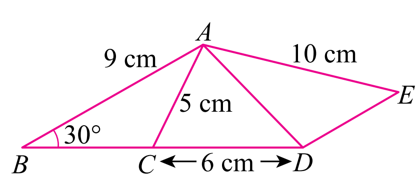
1. (a) Given $\angle ABC = 50°$, $\angle BAC = 72°$ and $c = 5.8$ cm, calculate the length of a and b. PL1
   (b) Given the sides of triangle PQR are $p = 8.28$ cm, $q = 6.56$ cm and $r = 3.63$ cm, find $\angle P$, $\angle Q$ and $\angle R$. PL2

2. Find the value of x in each of the following diagrams. PL3
   (a)


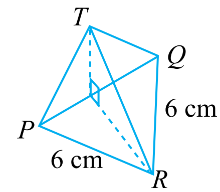
   (b)


3. The diagram on the right shows a right-angled triangle ABC. Point D lies on AB. Calculate PL3
   (a) the length of AC,
   (b) the area of triangle ADC.


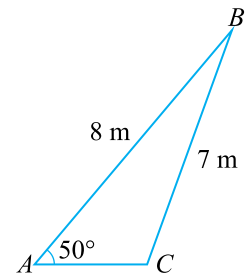
4. Given triangle XYZ such that $\angle X = 42.2°$, $x = 10$ cm and $z = 13.4$ cm. PL4
   (a) Sketch two possible shapes of the triangle.
   (b) Hence, find the possible values of $\angle Z$.
   (c) Calculate the area of triangle XYZ for the obtuse angle of $\angle Z$.

5. The diagram on the right shows five points, A, B, C, D and E which forms quadrilaterals. BCD is a straight line, $\angle ACB$ is obtuse and the area of triangle ADE is 20 cm^2. Calculate PL4
   (a) the length of AD,
   (b) $\angle DAE$.


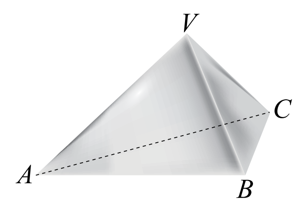
6. In the diagram on the right, PQR is an equilateral triangle with sides of 6 cm and lies on a horizontal plane. Point T is 4 cm vertically above the midpoint of PQ. Calculate PL5
(a) the angle formed by TR and triangle PQR,
(b) the area of plane TPR.



7. A group of girl scouts took part in a camping. They set up three tents with the positions as shown in the diagram on the right. The positions of the three tents formed a triangle ABC. PL5
(a) Calculate the obtuse angle ACB.
(b) Draw and label another triangle other than triangle ABC which shows the possible position of tent C such that the distance AB and AC and $\angle ABC$ remain unchanged.
(c) Tent C has to be relocated to other position but the distance between tent A and tent B and angle BAC formed between the tents remain unchanged. Calculate the distance AC such that only one triangle can be formed.

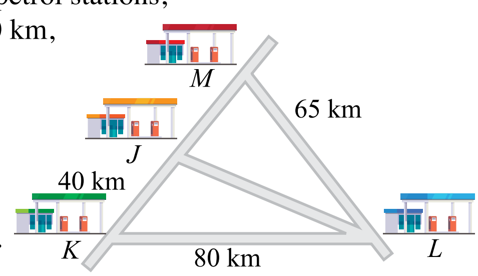


8. The diagram on the right shows a glass block in the shape of a pyramid $VABC$. The base of the block is an isosceles triangle and $AB = AC = 5.2$ cm. V is the vertex of the block such that $BV = CV = 3$ cm. The angle between the inclined plane VBC and the base ABC is 50°. Calculate PL5
(a) $\angle BAC$, given the base area is 8.69 cm^2,
(b) the length of AV, given the angle between the line AV and the base is 25°,
(c) the surface area of VAB of the glass block.



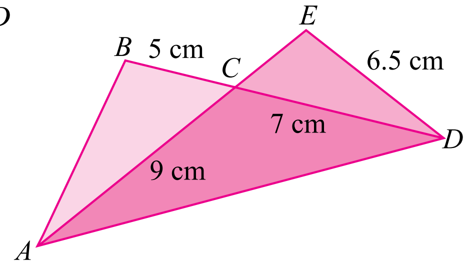
9. Rashid drove a boat westwards. He noticed a lighthouse at a distance of 25 km away at the bearing of 235°. PL5
(a) Sketch a diagram to illustrate the above situation.
(b) What is the distance travelled by the boat if its distance from the lighthouse is 16 km?
(c) Rashid continued to drive the boat until his distance from the lighthouse is 16 km again.
 (i) Calculate the distance between the first position and the second position of the boat.
 (ii) What is the bearing of the lighthouse from the boat when the boat is at the second position?

10. The diagram on the right shows the positions of four petrol stations, J, K, L and M in a district. Given the distance $JK = 40$ km, $KL = 80$ km, $LM = 65$ km and $\angle JKL = 44°$. PL5
(a) Calculate
 (i) the distance JL, (ii) $\angle JML$,
 (iii) the area of KLM.
(b) Without doing calculations, determine the petrol station which is the furthest from petrol station K. Explain.
(c) If a car travels along the road KL, calculate the shortest distance of the car from petrol station M.



CHAPTER 9

11. Mary coloured the three triangles, ABC, ACD and CED such that ACE and BCD are straight lines. Given $\angle DCE = 50.05°$ and $\angle CED$ is obtuse. PL6

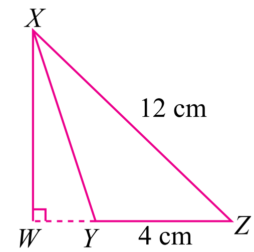


(a) Calculate
   (i) $\angle CED$,
   (ii) the length of AB,
   (iii) the area of triangle AED.

(b) The straight line AB is extended to point B' such that $CB' = CB$. On the same diagram, draw and colour the triangle BCB'.

12. In the diagram on the right, WYZ is a straight line.

Given $\sin \angle XYW = \dfrac{10}{11}$. PL6



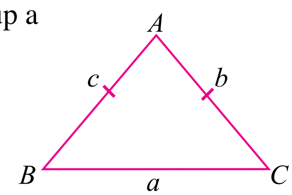
(a) Find $\sin \angle YXZ$.

(b) Calculate the area of triangle XYZ. Hence, find the length of XW.

(c) State two situations so that the ambiguous case exists in the triangle on the right.

Exploring MATHEMATICS

You are given a roll of wire of 100 metres length. You have to fence up a region in the shape of an isosceles triangle. The diagram on the right shows the sketch of the triangular region.



(a) Complete the following table to find the possible lengths of sides, a, b and c, of the triangle that can be formed by using the wire.

a	b	c	Area of triangle
2	49	49	
4	48	48	

(b) By using suitable formulae and technology, calculate the area of each triangle.

(c) Hence, predict the maximum area which can be fenced up and state the shape of the triangle.

CHAPTER 10 Index Numbers

What will be learnt?

- Index Numbers
- Composite Index

List of Learning Standards

bit.ly/2q4n8El

KEYWORDS

Index number	*Nombor indeks*
Price index	*Indeks harga*
Quantity at base time	*Kuantiti pada masa asas*
Quantity at specific time	*Kuantiti pada masa tertentu*
Composite index	*Indeks gubahan*
Weightage	*Pemberat*

Consumer Price Index (CPI) measures the changes in the prices of goods and services that represent the average purchasing pattern of a group of people over a period of time. CPI is also used to calculate the inflation rate and cost of living. Beside food and drinks, what are other goods and services that can be bought by Malaysian households?

Did you Know?

In the year 1764, Giovanni Rinaldo Carli (1720-1795) who was an Italian economist calculated the price ratios of three goods from the year 1500 to 1750. The average of the price ratios of the three goods represented the measures of change that had happened in the period of 250 years. His idea has resulted in the extended usage of index number till today.

For further information:

bit.ly/33ngFCU

SIGNIFICANCE OF THIS CHAPTER

In general, index number is used to measure all types of quantitative changes in the field of industries, agricultures, trades and services. Besides these, index number also plays an important role in measuring the magnitude of the economy such as income, job opportunity, export, import, price and others.

Scan this QR code to watch a video on Consumer Price Index (CPI) in Malaysia.

bit.ly/2PvIIMt

10.1 Index Numbers

Defining and describing index numbers

Based on the conversation above, what conclusion can be made regarding the prices of the smart phones in this year and last year? If you are able to state an increase of 10% in the prices, then you have made a relation regarding the index number.

In general, index number is a statistical measure to measure the change of a variable of a particular year as compared to another year which is considered to be the base year. The base usually has the value of 100 and index number is 100 times the ratio with the base. The variables can consist of value of currency, price, product, earning, quantity, job opportunity and others.

Mathematics Museum

The earliest recorded calculation of index number was in 1750.

There are varieties of index numbers with different calculations. For example:

Consumer price index

$$IHP = \frac{\text{Market price at the current year}}{\text{Market price at base year}} \times 100$$

Death index of road accidents

$$I = \frac{a}{\sum \text{vehicles}} \times 10\ 000$$

a = total death at the current year
$\sum$vehicles = total number of registered vehicles till the current year

Air quality index

$$I = \frac{I_{high} - I_{low}}{C_{high} - C_{low}}(C - C_{low}) + I_{high}$$

I = Air quality index
C = Concentration of pollutants

Body mass index

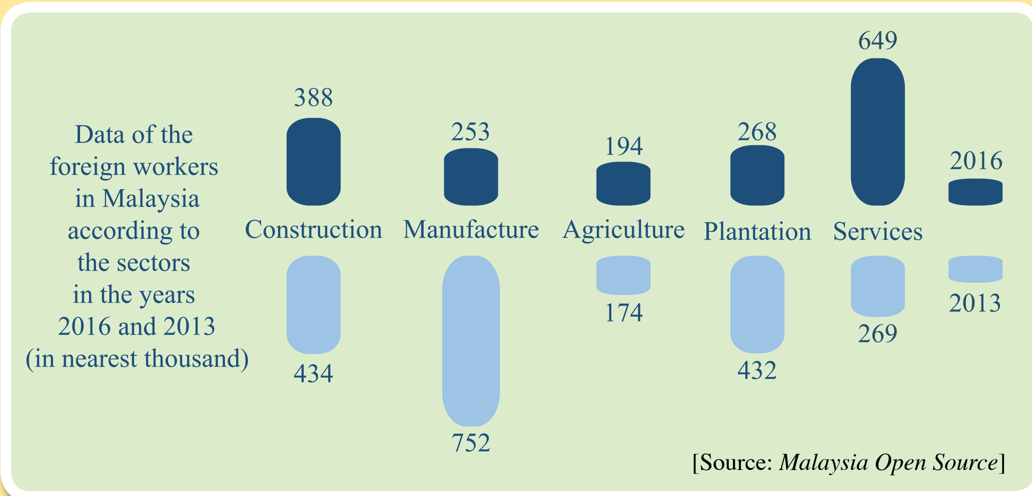
$$\text{BMI} = \frac{\text{Weight (kg)}}{\text{Height (cm)} \times \text{Height (cm)}} \times 100$$

INQUIRY 1 In groups 21st Century Learning

Aim: To determine the percentage change and relate with index numbers

Instruction:

1. Work in 5 groups.
2. Look at the graphical info below regarding the data of total foreign workers according to the sectors in Malaysia in the years 2013 and 2016.



3. Every group is required to pick only one sector to analyse.
4. In groups, answer the following questions:
   (a) Determine the percentage change in the data for the year 2016 compared to that of the year 2013 for each sector and interpret the percentage change obtained.
   (b) List down the causes for the changes to take place.
   (c) State two implications of the entry of foreign workers to our country.
   (d) List down suggestions to overcome the negative effects of entry of the foreign workers in this country.
5. Present your work in a creative way in front of the class.
6. Carry out a question and answer session with the members of other groups.

From the results of Inquiry 1, with the year 2013 as the base year, the index number is the percentage change of the data of the foreign workers in Malaysia in the year 2016 compared to the year 2013.

Percentage change in data of foreign workers in construction sector $= \frac{388}{434} \times 100\%$
$= 89.4\%$

The above percentage can also be written in index number, I:

$$I = \frac{388}{434} \times 100$$
$$= 89.4$$

10.1.1 10.1.2

In general, the formula for index number can be written as:

$$I = \frac{Q_1}{Q_0} \times 100$$

with Q_0 = Price/Quantity at the base year
Q_1 = Price/Quantity at a particular year

MATHEMATICS POCKET

Price index or quantity is the ratio in percentage, without writing the percentage symbol.

Example 1

The price of an X-branded watch in the year 2017 and 2018 was RM500 and RM550 respectively. Calculate the index number of the price of the watch in the year 2018 based on the year 2017. Interpret the index number obtained.

Solution

Let Q_0 = Price in the year 2017
Q_1 = Price in the year 2018

$$\text{Index number, } I = \frac{Q_1}{Q_0} \times 100$$
$$= \frac{550}{500} \times 100$$
$$= 110$$

Thus, there is an increase of 10% in the price from the year 2017 to the year 2018.

Smart TIPS

The value of index number more than 100 means there will be an increase when compared to the base year whereas index number smaller than 100 means there will be a decrease or reduction when compared to the base year.

Example 2

In the year 2017, the number of sports governing bodies registered with the Sports Commissioner Office is 893. Given that the index number of the registration of sports governing bodies of the year 2017 based on the year 2010 is 156.39, calculate the number of sports governing bodies registered in the year 2010.

Solution

Let Q_0 = The number of registration in the year 2010
Q_1 = The number of registration in the year 2017

$$I = \frac{Q_1}{Q_0} \times 100$$
$$156.39 = \frac{893}{Q_0} \times 100$$
$$Q_0 = 571$$

Thus, the number of sports governing bodies registered in the year 2010 is 571.

Mind Challenge

Can the value of index number be 100? If so, when will this arise?

10.1.1 10.1.2

Example 3

The price index of a bicycle in the year 2018 based on the year 2010 and 2015 was 176 and 110 respectively. Find the price index of the bicycle in the year 2015 based on the year 2010.

Solution

$$\frac{Q_{2018}}{Q_{2010}} \times 100 = 176 \quad \cdots ①$$

$$\frac{Q_{2018}}{Q_{2015}} \times 100 = 110 \quad \cdots ②$$

$$① \div ②: \quad \frac{Q_{2015}}{Q_{2010}} = \frac{176}{110}$$

$$I = \frac{Q_{2015}}{Q_{2010}} \times 100$$

$$= \frac{176}{110} \times 100$$

$$= 160$$

Alternative Method

$$I_{2018/2015} = \frac{I_{2018/2010}}{I_{2015/2010}} \times 100$$

$$110 = \frac{176}{I_{2015/2010}} \times 100$$

$$I_{2015/2010} = \frac{176}{110} \times 100$$

$$= 160$$

QR

Scan this QR code for other methods to solve problems involving index numbers.

bit.ly/2Ntoh02

Self Practice 10.1

1. Malaysia Automotive Association (MAA) reported that the total number of registered commercial vehicles in the year 2015 was 75 376 whereas the total registered number commercial vehicles in the year 2017 was 61 956. Calculate the index for the number of registered commercial vehicles in the year 2017 based on the year 2015 and interpret it.
2. The average monthly expenditure of a Malaysian household in the year 2014 was RM3 578. In the year 2017, average monthly expenditure of a Malaysian household was RM4 033. Find the average index for the monthly expenditure of a Malaysian household in the year 2017 based on the year 2014 and interpret your findings.
3. The total production of oil palm in Malaysia in the year 2013 was 720 440 105 metric tonnes. Given the index of the total production of oil palm in the year 2016 based on the year 2013 was 90.23, find the total production of oil palm in the year 2016.
4. The table below shows the price indices for a particular type of drink.

Year 2013 (2011 = 100)	Year 2019 (2011 = 100)	Year 2019 (2013 = 100)
150	225	p

Find the value of p.

Smart TIPS

The year 2013 (2011 = 100) means price index in the year 2013 is based on the year 2011.

5. The production index of industrial sugar in the year 2011 and 2012 based on the year 2010 was 101.4 and 95.8 respectively. Calculate the index production of industrial sugar in the year 2012 based on the year 2011.

10.1.1 10.1.2

Problem solving involving index numbers

Example 4 MATHEMATICS APPLICATION

According to the statistics from the Ministry of Natural Resources and Environment, the total number of visitors who visited Taman Negara Pahang, Sungai Relau in the year 2016 was 17 721. If National Park Corporation targeted a 10% increase in the number of visitors for the year 2018, calculate the expected number of visitors in the year 2020 if the rate of increment of the visitors from the year 2018 to the year 2020 is the same as the rate of increment of the visitors from the year 2016 to the year 2018.

Solution

1. Understanding the problem

- Number of visitors in the year 2016 was 17 721.
- The increment of 10 percent from the year 2016 to the year 2018.
- The increment of 10 percent from the year 2018 to the year 2020.
- Find the number of visitors in the year 2020.

2. Planning the Strategy

- Find the number of visitors in the year 2018 by using the index number formula.
- By using the number of visitors in the year 2018, the number of visitors in the year 2020 is calculated using the index number formula.

3. Implementing the strategy

The number of visitors in the year 2018

$$I_{2018/2016} = \frac{Q_{2018}}{Q_{2016}} \times 100$$

$$110 = \frac{Q_{2018}}{17\ 721} \times 100$$

$$Q_{2018} = 19\ 493$$

The number of visitors in the year 2020

$$I_{2020/2018} = \frac{Q_{2020}}{Q_{2018}} \times 100$$

$$110 = \frac{Q_{2020}}{19\ 493} \times 100$$

$$Q_{2020} = 21\ 442$$

Thus, the expected number of visitors in the year 2020 is 21 442.

4. Making a conclusion

- The index number in the year 2020 is based on the year 2018, $\frac{21\ 442}{19\ 493} \times 100 \approx 110$
- The index number in the year 2018 is based on the year 2016, $\frac{19\ 493}{17\ 721} \times 100 \approx 110$

CHAPTER 10

10.1.3

INQUIRY 2 In groups 21st Century Learning

Aim: To study the usage of index number
Instruction:
1. Read the newspaper extract below carefully.

The alarming rate of accidents among people

BANGI: National Institute of Occupational Safety and Health (NIOSH) expressed their concerns over the alarming rate of accidents among people in the country from 66 618 cases in the year 2016 to 69 980 cases in the year 2017.

The chairman of NIOSH, Tan Sri Lee Lam Thye said, according to the statistics released by Social Security Organisation (SOCSO), as many as 33 319 cases were recorded in 2017 involving accidents which occurred when travelling to or returning from workplaces, the increase was as much as 6.4% from 31 314 cases of accidents recorded in the year 2016. On the other hand, the rate of occupational hazards increased as much as 3.84% from 35 304 cases in the year 2016 to 36 661 cases in the year 2017.

"This increase is particularly worrying as we celebrated our 61st Independence Day, we are only independent or free from colonial shackles, but still not independent from the aspect of attitude especially on the road," he said in the media conference after officiating celebration of the 61st Independence Day 2018 at NIOSH headquarters.

Translated from Berita Harian
(Source: https://www.bharian.com.my/berita/nasional/2018/08/468225/kadar-kemalangan-di-kalangan-rakyat-membimbangkan)

2. Carry out a brainstorming session among group members and answer the following questions:
   (a) Make a conjecture regarding the occupational hazard index in the year 2017 compared to the year 2016.
   (b) What are the effects if the rate of occupational hazards continue to increase?
   (c) What are the causes of the increase of occupational hazards in our country?
   (d) Suggest ways to reduce the rate of occupational hazards in our country.
3. Prepare a graphical folio to answer the above questions.
4. Display your group result to the class.

Self Practice 10.2

1. The table shows the price index of groceries in the year 2015 and 2020 based on the year 2010.

Item	Price index in the year	
	2015	2020
Groceries	125	140

Find the price index of these groceries in the year 2020 based on the year 2015.

2. The premium insurance payment of a company in the year 2016 increased as much as 5 percent compared to the year 2011. In the year 2018, the premium increased again as much as 10 percent compared to 2011. Find the premium insurance index in the year 2018 compared to the year 2016.

10.1.3

Intensive Practice 10.1

Scan the QR code or visit bit.ly/2lxHTyD for the quiz

1. In January 2017, the average temperature in town *P* was 25.3°C whereas the average temperature in February 2017 was 27.4°C. Find the average temperature index in February with January as the base time and interpret the index number obtained.

2. Given the price index of a certain item in the year 2016 based on the year 2015 was 130 and the price index in the year 2016 based on the year 2012 was 120. Find the price index of this item in the year 2015 based on the year 2012 and interpret it.

3. The table below shows the prices and the price indices of three ingredients *P*, *Q* and *R* which are used in preparing a type of biscuit.

Material	Cost (RM/kg)		Price index in the year 2019 based on the year 2015
	Year 2015	Year 2019	
P	x	0.40	80
Q	2.00	y	140
R	0.80	1.00	z

Find the values of x, y and z.

4. The table below shows the retail prices of a chicken in January for the year 2015 until 2018.

Year	Price (RM/kg)	Price index
2015	5.80	p
2016	7.65	q
2017	7.80	r
2018	7.30	s

Using the year 2015 as the base year, find the value of p, q, r and s.

5. The diagram below shows the price index for a type of food in 2015 and 2018 based on the year 2010.

Item	Price index	
	Year 2015	Year 2018
Food	110	118

Find the price index of the food in the year 2018 based on the year 2015.

10.2 Composite Index

Determining and interpreting composite index

INQUIRY 3 In pairs 21st Century Learning

Aim: Determine composite index

Instruction:

1. The table below shows the price indices and percentages of four ingredients used in preparing *semperit* in the year 2019 based on the year 2018.

Ingredient	Price index	Percentage (%)
Margarine	120	30
Sugar	127	15
Wheat flour	108	50
Egg	107	5

2. Calculate the average price index for all the four ingredients and make a conclusion about the average value.
3. What is the role played by the value of percentages in the calculation of the average price index? If the percentages of these four ingredients are the same, what can you interpret about the average price index?
4. Present your findings in front of the class and carry out a question and answer session with other pairs.

From Inquiry 3, the average price index is obtained as follow:

$$\text{Average price index} = \frac{(120 \times 30) + (127 \times 15) + (108 \times 50) + (107 \times 5)}{100}$$
$$= 114.4$$

The average price index indicates that there is an increase in the price of the raw ingredients in the year 2019 compared to the year 2018. The value of the percentages represents the importance of the usage of the raw materials in the preparation of *semperit*.

The value of this average price index is known as the **composite price index** ($\bar{I}$) which means the combination of a few indices as a statistical measure for overseeing the market or sector performance from time to time involving the importance of each item. The importance is known as the **weightage** (w). The value of weightage can be in the form of numbers, ratios, percentage, readings on the bar chart or pie chart and others.

If $I_1, I_2, I_3, \ldots, I_n$ are the price indices for n items respectively with weightages $w_1, w_2, w_3, \ldots, w_n$, then the composite index can be calculated using the following formula:

$$\bar{I} = \frac{(I_1w_1 + I_2w_2 + I_3w_3 + \ldots + I_nw_n)}{w_1 + w_2 + w_3 + \ldots + w_n}$$

$$\bar{I} = \frac{\sum I_iw_i}{\sum w_i}$$

with I_i = index numbers and w_i = weightages

10.2.1

Example 5

Price index of one kilogram of three types of fruits sold in a stall in the year 2018 based on the year 2010 was 175, 120 and 160 respectively. Find the composite index of these fruits in the year 2018 based on the year 2010.

Solution

Composite index, $\bar{I} = \dfrac{\sum I_i w_i}{\sum w_i}$

$\bar{I} = \dfrac{175(1) + 120(1) + 160(1)}{3}$ ← the weightage for each type of fruits is 1

$= 151.67$

MATHEMATICS POCKET

Composite index without weightage given is calculated by assuming the value of the weightages are the same for each index number.

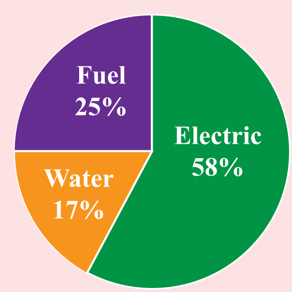
Mind Challenge

What is the difference between composite index with and without weightages given? Explain the importance of weightage in the calculation of composite index.

Example 6

The table below shows the expenditure of utility index of a factory in the year 2017 based on the year 2011. The pie chart shows the percentages of the usage in a month.

Utility	Expenditure index
Water	135
Electricity	140
Fuel	125



Find the composite index of the expenditure of utility in the year 2017 based on the year 2011.

Solution

Composite index, $\bar{I} = \dfrac{\sum I_i w_i}{\sum w_i}$

$= \dfrac{135(17) + 140(58) + 125(25)}{17 + 58 + 25}$

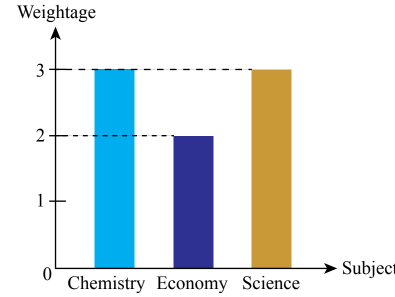
$= \dfrac{13\,540}{100}$

$= 135.4$

Self Practice 10.3

1. The price index of the traditional *kuih* such as *nekbat*, *nagasari* and *serabai* in the year 2020 based on the year 2015 is 105, 112 and 98 respectively. Find the composite index for the three types of traditional *kuih* in the year 2020 based on the year 2015 and interpret the value obtained.

2. The bar chart shows the credit points of three subjects in a college. Given the student entry index following the subjects Chemistry, Economics and Science in the year 2019 based on the year 2015 was 136, m and 108 respectively. Find the value of m if the composite index for the three subjects in the year 2019 based on the year 2015 was 120.



Solving problem involving index numbers and composite numbers

The concepts of index numbers and composite numbers studied this far are used widely in various fields for recognising and monitoring the trend in prices, production, job opportunities, inflation and others.

Example 7

The table below shows the cost price of three main materials in making non-rusting steel by a company.

Material	Price in the year 2010 (RM per metric tons)	Price in the year 2018 (RM per metric tons)	Percentage (%)
Iron	2 025	3 424	72
Chromium	8 431	9 512	18
Nickel	62 235	50 916	10

(a) Calculate the price index for iron, chromium and nickel in the year 2018 based on the year 2010.
(b) Calculate the composite index for the cost of the production of the non-rusting steel in the year 2018 based on the year 2010. Interpret your findings.
(c) Determine the cost of production of the non-rusting steel in the year 2018 if the cost in the year 2010 was RM65 million.

Solution

(a) $I_{\text{Iron}} = \dfrac{Q_{2018}}{Q_{2010}} \times 100 = \dfrac{3\,424}{2\,025} \times 100 = 169.09$

$I_{\text{Chromium}} = \dfrac{Q_{2018}}{Q_{2010}} \times 100 = \dfrac{9\,512}{8\,431} \times 100 = 112.82$

$I_{\text{Nickel}} = \dfrac{Q_{2018}}{Q_{2010}} \times 100 = \dfrac{50\,916}{62\,235} \times 100 = 81.81$

Thus, the price index for iron, chromium and nickel in the year 2018 based on the year 2010 was 169.09, 112.82 and 81.8 respectively.

CHAPTER 10

10.2.1 10.2.2

(b) Construct a table to determine Σw_i and $\Sigma I_i w_i$.

Material	I_i	w_i	$I_i w_i$
Iron	169.09	72	12 174.48
Chromium	112.82	18	2 030.76
Nickel	81.81	10	818.10
		$\Sigma w_i = 100$	$\Sigma I_i w_i = 15\ 023.34$

$$\bar{I} = \frac{\Sigma I_i w_i}{w_i}$$
$$= \frac{15\ 023.34}{100}$$
$$= 150.23$$

There was an increase of 50.23% in the production cost of the non-rusting steel in the year 2018 based on the year 2010.

(c) $$I = \frac{Q_{2018}}{Q_{2010}} \times 100$$
$$150.23 = \frac{Q_{2018}}{65} \times 100$$
$$Q_{2018} = 97.65$$

Thus, the production cost of the non-rusting steel in the year 2018 was RM97.65 million.

BRAINSTORMING

Did the decreasing price of the nickel in 2018 affect the total production cost of the production of the non-rusting steel? Discuss.

Self Practice 10.4

1. The table below shows the price of four materials, A, B, C and D used in making roof tiles in the year 2016 and 2010.

Material	Cost (RM)		Weightage (%)
	2010	2016	
A	1.40	2.10	10
B	1.50	1.56	20
C	1.60	1.92	40
D	4.50	5.58	30

(a) Calculate the price index for each material in the year 2016 based on the year 2010.
(b) Calculate the composite index for all the materials in the year 2016 based on the year 2010. Interpret your findings.
(c) Determine the price of the roof tile in the year 2010 if the price was RM2.65 in the year 2016.

2. The table below shows the price of five materials used in making a souvenir in the year 2013 and 2019.

Material	Price in the year 2013 (RM)	Price in the year 2019 (RM)	Price index (2013 = 100)	Weightage (%)
P	5.00	6.00	120	8
Q	20.00	23.00	a	12
R	8.00	12.00	b	20
S	16.00	18.00	c	27
T	10.00	13.00	130	d

(a) Calculate the value of a, b, c and d.
(b) Calculate the composite index for the souvenir in the year 2019 based on the year 2013. Interpret your findings.
(c) Determine the price of the souvenir in the year 2019 if the price in the year 2013 was RM35.
(d) Calculate the price index of the souvenir in the year 2021 if the total cost of the materials used is expected to increase by 10% in the year 2021.

Intensive Practice 10.2

Scan the QR code or visit bit.ly/2Ox3nPM for the quiz

1. The admission of students in a school according to the Science stream and the Arts stream follows the ratio 60 : 40. Given that the admission index of students according to the Science stream and the Arts stream in the year 2019 based on the year 2015 was 120 and 130 respectively. Find the composite index for the admission of students in the school in the year based on the year 2019 and 2015.

2. Myra Company has three small subsidiaries in three districts in Selangor. The table below shows the change in productivity and the number of workers in the three subsidiaries in the year based on the year 2018 and 2010.

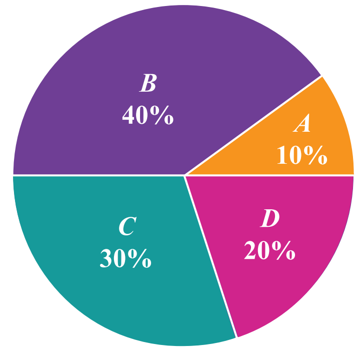
District	Change in productivity from the year 2010 to 2018	Number of workers
Kuala Langat	Increased 10%	3
Gombak	No change	2
Shah Alam	Decreased 20%	5

Find the composite index for the productivity of the three subsidiaries. Give your opinion regarding the productivity of Myra Company based on the value you have obtained.

3. The subject evaluation in a college follows the Paper 1, Paper 2 and Course Work format. The allocation for Course Work is 20% of the total marks of the subject whereas the marks for Paper 1 and Paper 2 is 80% of the total marks and both of the papers are important in the calculation of the final marks. Kalaivathy obtains 85, 72 and 68 marks for Paper 1, Paper 2 and Course Work respectively. Calculate the final marks obtained by Kalaivathy in the subject.

4. The table below shows the price index and change in the price indices for four main materials in the production of a facial wash.

Material	Price index in the year 2021 based on the year 2019	Change in the price index from the year 2021 to the year 2023
A	150	No change
B	140	Reduced by 10%
C	*m*	No change
D	115	Increased by 20%

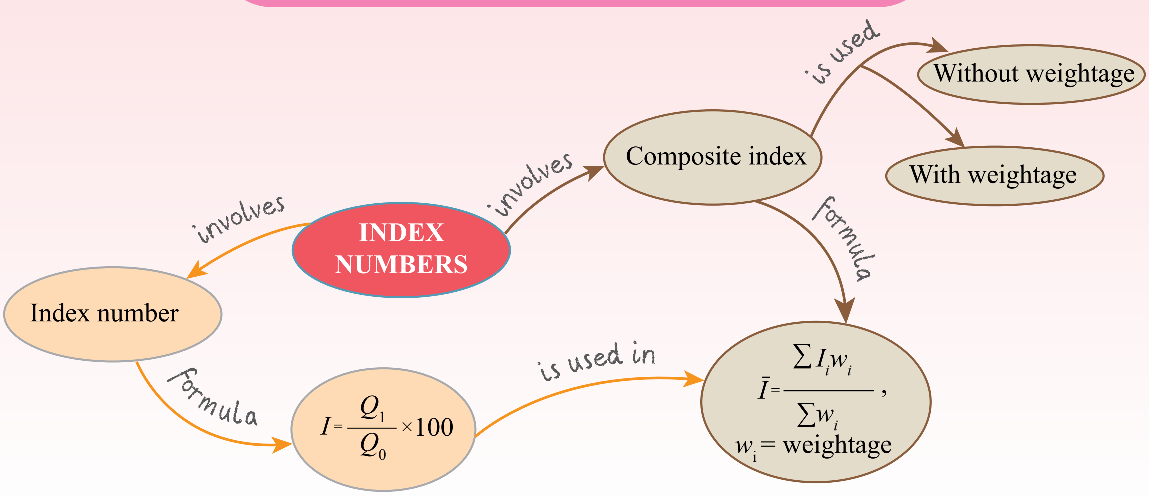


(a) Find the value of *m* if the composite index in the production of facial wash in the year 2021 based on the year 2019 is 133.

(b) Calculate the composite index in the production of the facial wash in the year 2023 based on the year 2019.

(c) Calculate the production cost of the facial wash in the year 2023 if the cost corresponds to the year 2019 is RM19.50.

SUMMARY OF CHAPTER 10



WRITE YOUR JOURNAL

Based on your comprehension throughout this chapter, what do you understand about index number? In your opinion, what is the best way to determine the most suitable base year in calculating the index number of goods or services? What is weightage? What are the factors that affect the relative importance of an item?

MASTERY PRACTICE

1. The table below shows the price per kg of four types of goods, *A*, *B*, *C*, and *D*, in the year 2017 and 2019, price index in the year 2019 was based on the year 2017 and their weightages respectively. PL3

Good	Price in the year 2017 (RM/kg)	Price in the year 2019 (RM/kg)	Price index in the year 2019 (2017 = 100)	Weightage
A	2.00	2.20	z	4
B	0.80	y	125	1
C	1.10	1.10	100	2
D	x	1.20	120	3

(a) Find the value of x, y and z.
(b) Calculate the composite index of the goods in the year 2019 based on the year 2017.

2. The table below shows the price index of two materials *A* and *B* used in production of a type of household decorations. PL3

Material	Price index in the year 2018 based on the year 2016	Price index in the year 2020 based on the year 2016
A	110	m
B	n	110

Given the price of material *B* increased by 22% in the year 2018 from the year 2016. The price of material *A* in the year 2016 was RM5.00 and the year 2020, RM6.05. Find the value of m and n.

3. The table below shows the information related to four materials, *A*, *B*, *C* and *D* used in making a toy. The percentage of usage of material *B* is not shown. PL4

Material	Change in price index from the year 2015 to the year 2018	Percentage usage (%)
A	Reduced by 10%	50
B	Increased 60%	
C	Increased 20%	10
D	Increased 40%	10

The production cost of the toy was RM41 650 in the year 2018.
(a) If the cost of material *C* in the year 2015 was RM7.60, find the cost in the year 2018.
(b) Calculate the corresponding production cost in the year 2015.
(c) The production cost is expected to increase by 60% from the year 2018 to the year 2020. Calculate the percentage change in the production cost from the year 2015 to the year 2020.

4. The rubber production in Malaysia is 1.126 million tonnes in 2005, x million tonnes in 2010 and 0.722 million tonnes in 2015. Calculate PL3
   (a) the index number for the rubber production in the year 2015 based on the year 2005,
   (b) the value of x, given that the index number for rubber production in the year 2010 based on the year 2005 is 83,
   (c) the index for the rubber production in the year 2020 based on the year 2005 if the index for rubber production in the year 2020 based on the year 2010 is 105.

5. The table below shows the price for an item in 2000 and 2015. PL4

Year	Price
2000	RM8
2015	RM10

   (a) If the rate of price increase from 2015 to 2020 is twice the rate of price increase from 2000 to 2015, determine the price of that item in 2020.
   (b) Calculate the price index in the year 2020 based on the year 2000.

6. The table below shows the price index and weightage for four types of materials in the year 2020 based on the year 2019. PL4

Materials	Price index	Weightage
P	107	2
Q	118	x
R	94	1
S	105	$2x$

   (a) The composite index for the materials in the year 2020 based on the year 2019 is 108. Determine the value of x.
   (b) The price index for material P rises by 20% and the price index for material S drops by 10% in 2020 until 2021. The price index for other materials do not change. Determine the composite index for the materials in the year 2021 based on the year 2019.

7. The table below shows the sales index for an encyclopedia in 2015 and 2017 with the year 2000 as the base year. PL4

Year	2015	2017
Sales index	109	145

   Determine the sales index for the encyclopedia in the year 2017 based on the year 2015.

8. The table below shows the price index for three cameras. PL4

Year / Camera	2013 (2011 = 100)	2019 (2011 = 100)	2019 (2013 = 100)
J	165	231	p
K	q	156	120
L	150	r	170

   Determine the value of p, q and r.

CHAPTER 10

9. The following shows the number of visitors who visited Pulau Langkawi in 2010 and 2017. PL5

2010 2.45 million　　**2017** 3.68 million

(a) Determine the number of visitors in 2020 if the rate of increase for the number of visitors in 2017 to 2020 is twice the rate of increase from 2010 to 2017.
(b) Calculate the index for number of visitors in the year 2020 based on the year 2017. State your interpretation based on the index number obtained.

10. The table below shows the price and weightage for three materials P, Q and R in the year 2018 based on the year 2016. PL4

Material	P	Q	R
Price index	80	130	140
Weightage	x	y	z

Given the composite index for material P and Q in the year 2018 based on the year 2016 is 120 whereas the composite index for materials P and R is 125. Determine the ratio $x : y : z$.

11. The price index for a safety helmet in the year 2014 based on the year 2010 was 80 and the price index in the year 2018 based on the year 2014 was 110. Given the price for the safety helmet in the year 2018 was RM166. PL5
(a) Calculate the price of the safety helmet in 2010 and 2014.
(b) Determine the percentage of decrease in price for safety helmets in 2010 compared to its price in 2018.

12. The price for service charge in an agency in 2018 was RM1.50. If the price increased by 15% in 2019, calculate PL5
(a) the price index for service charge in 2019 with 2018 as the base year,
(b) the price for the service charge in 2020 if the rate of price increase in 2019 to 2020 is the same as the rate of price increase for 2018 to 2019.

Exploring MATHEMATICS

1. Prepare a monthly expenditure of your family according to each of the following categories for the span of 3 months.
(a) Food and beverages
(b) Clothing and shoes
(c) Water and electricity bills
(d) Transportation
(e) Medication
(f) Education

2. Explain the weightage based on the relative money spent by your family.

3. Determine the composite index for expenditures on the 2nd and 3rd month based on the 1st month. What conclusion can you make based on the composite index value obtained?

4. Explain ways to spend wisely.

5. Discuss in groups and make an interesting graphic folio.

ANSWERS

Open the complete answer file in the QR Code on page vii to get the full solutions

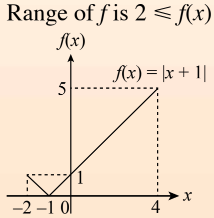
CHAPTER 1 FUNCTIONS

Self Practice 1.1

1. (a) Function because every object has only one image even though element 7 has no object.
   (b) Function because every object has only one image even though element 4 has two objects.
   (c) Not a function because object r has two images, 8 and 10.
2. (a) Function (b) Not a function
   (c) Function
3. (a) $h : x \to \frac{1}{x}, x \neq 0$ (b) $h : x \to |x|$
   (c) $h : x \to x^3$

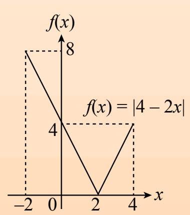
Self Practice 1.2

1. (a) Domain = $\{-2, -1, 0, 2, 4\}$
   Codomain = $\{1, 3, 4, 5\}$
   Range = $\{1, 3, 4, 5\}$
   (b) Domain = $\{j, k, l, m\}$
   Codomain = $\{2, 3, 6, 7, 10\}$
   Range = $\{3, 7\}$
   (c) Domain of f is $-3 \leqslant x \leqslant 5$
   Codomain of f is $2 \leqslant f(x) \leqslant 6$
   Range of f is $2 \leqslant f(x) \leqslant 6$
2. (a)



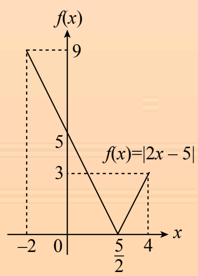
Range of f is $0 \leqslant f(x) \leqslant 5$.

(b)



Range of f is $0 \leqslant f(x) \leqslant 8$.

(c)



Range of f is $0 \leqslant f(x) \leqslant 9$.

Self Practice 1.3

1. (a) $g(-5) = 2, g(-2) = 1, g\left(\frac{1}{2}\right) = -9$
   (b) $b = -\frac{1}{2}, b = 3$
2. (a) $k = 4$ (b) $k = 3$
   (c) $k = 3$
3. (a) $f(-2) = 11, f\left(-\frac{1}{2}\right) = 5$ (b) $\frac{1}{2}, 1$
   (c) $\frac{1}{2} < x < 1$ (d) $x < -\frac{1}{2}, x > 2$
4. 2, 6
5. (a) $m = -4, c = 15$
   (b) 7 (c) 3

Intensive Practice 1.1

1. (a) and (c) because every object has only one image.
2. (a) Not a function (b) Not a function
   (c) Function
3. (a) Function because every object has only one image.
   (b) Domain = $\{-7, -6, 6, 7\}$
   Range = $\{36, 49\}$
   (c) $f : x \to x^2$
4. (a) $t = 6$ (b) $0 \leqslant f(x) \leqslant 6$
   (c) $0 \leqslant x \leqslant 4$
5. (a) (i) 80 meter (ii) 72 meter
   (iii) 45 meter
   (b) 3 seconds

Self Practice 1.4

1. (a) $f(x) = 3x$ (b) $gf(x) = 2x - 7$
2. (a) $fg : x \to 9 - 3x, gf : x \to 3 - 3x$
   $f^2 : x \to 9x, g^2 : x \to x$
   (b) $fg : x \to 4 + 2x^2, gf : x \to 4x^2 + 16x + 16$
   $f^2 : x \to 4x + 12, g^2 : x \to x^4$
   (c) $fg : x \to \frac{6}{x} + 4, x \neq 0, gf : x \to \frac{6}{x + 4}, x \neq -4$
   $f^2 : x \to x + 8, g^2 : x \to x$
   (d) $fg : x \to \frac{6 - 5x}{x - 1}, x \neq 1, gf : x \to \frac{1}{x - 6}, x \neq 6$
   $f^2 : x \to x - 10, g^2 : x \to \frac{x - 1}{2 - x}, x \neq 2$
3. $fg(x) = 3x^2 + 22, gf(x) = 9x^2 + 24x + 22$
   (a) $x = 1, x = 2$ (b) $x = 0, x = -4$
4. $a = -2, b = 9$ or $a = 2, b = -3$
5. $h = -k$

Self Practice 1.5

1. (a) $fg(3) = 4$ (b) $gf\left(-\frac{1}{5}\right) = 9$

(c) $f^2(4) = 3, g^2\left(\frac{1}{2}\right) = -1$

(d) $f^2(-1) = 5, g^2(-1) = -\frac{1}{2}$

2. (a) $x = 2$ (b) $x = 2, x = -2$
(c) $x = 2$ (d) $x = 1$

Self Practice 1.6

1. (a) $g : x \to 2x^2 - 4x + 10$
(b) $g : x \to x + 2$

2. (a) $g : x \to x^2 - 4x$ (b) $g : x \to 2x - 3$

3. (a) $g : x \to \frac{2}{x}, x \neq 0$ (b) $x = 24$

4. (a) $f(x) = 3x - 7$ (b) $gf(2) = -3$

Self Practice 1.7

1. (a) $f^2(x) = \frac{x}{2x + 1}, x \neq -\frac{1}{2}$

$f^3(x) = \frac{x}{3x + 1}, x \neq -\frac{1}{3}$

$f^4(x) = \frac{x}{4x + 1}, x \neq -\frac{1}{4}$

(b) $f^{20}(x) = \frac{x}{20x + 1}, x \neq -\frac{1}{20}$

$f^{23}(x) = \frac{x}{23x + 1}, x \neq -\frac{1}{23}$

2. (a) $f^2(x) = x$

$f^3(x) = \frac{1}{x}, x \neq 0$

$f^4(x) = x$

(b) $f^{40}(2) = 2$

$f^{43}(2) = \frac{1}{2}$

3. (a) $Ar(t) = \frac{16}{9}\pi t^6$ (b) $113\frac{7}{9}\pi \text{ m}^2$

4. (a) (i) $v(t) = 200 + 100t$ (ii) $h = \frac{v}{\pi r^2}$

(iii) $hv(t) = \frac{2 + t}{4\pi}$

(b) 1.75 cm

5. (a) $r(t) = 3t$
(b) $Ar(t)$ is the area of water ripple, in cm^2, as function of time, t in seconds
(c) $8100\pi \text{ cm}^2$

Intensive Practice 1.2

1. (a) $fg(x) = \frac{x - 1}{x + 1}, x \neq -1$

$gf(x) = \frac{2x - 1}{2x}, x \neq 0$

(b) $fg(2) = \frac{1}{3}$

$gf\left(-\frac{1}{2}\right) = 2$

(c) $x = \frac{1}{3}$

2. (a) $h = 3, k = -1$ (b) $\frac{5}{6}$

3. $a = 2, b = 3$

4. (a) $h = 2, k = -3$

(b) $gf(x) = \frac{-3x^2 + 18x - 19}{(x - 3)^2}, x \neq 3$

5. (a) $a = 3, b = 1$ (b) $f^4(x) = 81x + 40$

6. (a) $A(x) = x^2, V(A) = 10A$

7. (a) $g(x) = 2x^2 - 3x - 13$
(b) $g(x) = x^2 - 12x + 40$
(c) $g(x) = 14 - x$

8. (a) $g : x \to \frac{x - 1}{3}$ (b) $f(x) \to 9x^2 - 3x + 4$

9. (a) $p = 2, q = -1$ (b) $f^4(x) = 16x - 15$
(c) $f^n(x) = 2^n x + 1 - 2^n$

10. $CN(t) = 15\,000 + 800\,000t - 40\,000t^2$

Self Practice 1.8

1. (a) $f(4) = -5$ (b) $f^{-1}(-1) = 6$
(c) $f^{-1}(2) = -2$ (d) $f^{-1}(-5) = 4$

2. (a) $g(12) = -\frac{1}{2}$ (b) $g^{-1}(4) = \frac{3}{4}$

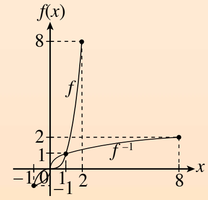
(c) $h(-1) = 3$ (d) $h^{-1}(9) = 1$

Self Practice 1.9

1. (a) Has an inverse
(b) Does not have an inverse
(c) Does not have an inverse
(d) Has an inverse
(e) Does not have an inverse
(f) Does not have an inverse
(g) Has an inverse

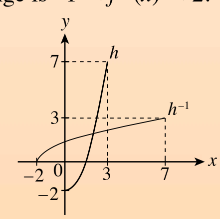
2. (a) Inverse function
(b) Inverse function
(c) Not an inverse function
(d) Not an inverse function

3.



The domain of function f^{-1} is $-1 \leqslant x \leqslant 8$ and the range is $-1 \leqslant f^{-1}(x) \leqslant 2$.

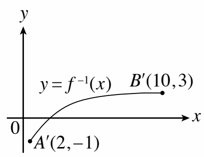
4. (a)



(b) The domain of function h^{-1} is $-2 \leqslant x \leqslant 7$
(c) $x = 2$

5. (a) $P'\left(\frac{1}{2}, -2\right)$ (b) $Q'(-3, 1)$
(c) $R'(5, 4)$ (d) $S'(-8, -6)$

6. (a)



(b) $a = 1, b = 4$

Self Practice 1.10

1. (a) $f^{-1} : x \to \dfrac{x+5}{2}$ (b) $f^{-1} : x \to \dfrac{3}{x}, x \neq 0$

(c) $f^{-1} : x \to \dfrac{4+x}{x}, x \neq 0$ (d) $f^{-1} : x \to \dfrac{6x}{x-5}, x \neq 5$

(e) $f^{-1} : x \to \dfrac{8x+9}{x-1}, x \neq 1$ (f) $f^{-1} : x \to \dfrac{x-3}{2x-2}, x \neq 1$

2. (a) $f^{-1}(4) = \dfrac{1}{3}$ (b) $x = -\dfrac{3}{2}, x = 1$

3. $a = -\dfrac{5}{2}, b = \dfrac{1}{8}$

4. (a) $f : x \to \dfrac{x-7}{6}$ (b) $f : x \to 2 - 5x$

(c) $f : x \to \dfrac{3x}{x-3}, x \neq 3$

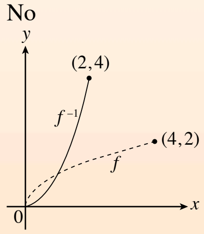
5. (a) $k = 2$ (b) $g\left(\dfrac{1}{2}\right) = -6$

Intensive Practice 1.3

1. (a) $f(2) = 5$ (b) $g(5) = 8$
(c) $gf(2) = 8$ (d) $f^{-1}(5) = 2$
(e) $g^{-1}(8) = 5$ (f) $f^{-1}g^{-1}(8) = 2$

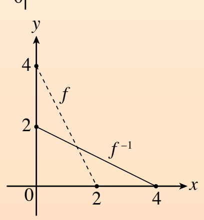
2. (a) Yes (b) Yes
(c) No

3. (a)



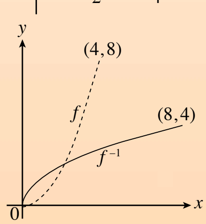
The domain of function f^{-1} is $0 \leqslant x \leqslant 2$.

(b)



The domain of function f^{-1} is $0 \leqslant x \leqslant 4$.

(c)



The domain of function f^{-1} is $0 \leqslant x \leqslant 8$.

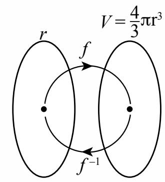
4. (a) $h = 5$ (b) $f^{-1}(3) = 14$
(c) $m = -9$

5. (a) $h(x) = \dfrac{3x-2}{x}, x \neq 0$ (b) $x = 2$

6. $x = -5, x = 2$

7. (a) $f^{-1}(x) = 220 - \dfrac{20}{17}x$ (b) 173.4

8. (a)



(b) 0.49 cm

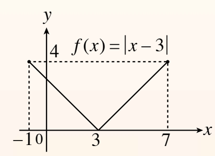
MASTERY PRACTICE

1. (a) (i) 1 (ii) 6, 8, 9
(b) Yes because every object only has one image.
(c) Domain = {2, 6, 7, 8, 9}
Codomain = {1, 4, 5}
Range = {1, 4}

2. (a) $m = 35$ (b) $h : x \to x^2 - 1$

3. Function but not a one-to-one function.

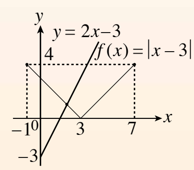
4. (a)



The range of function f is $0 \leqslant f(x) \leqslant 4$.

(b) $1 \leqslant x \leqslant 5$

(c)



$x = 2$

5. (a) $h = 7, k = 6$ (b) 43

6. $m = 3, c = -13$

7. (a) (i) $f^{-1}(x) = \dfrac{x+2}{3}$ (ii) $g(x) = 2x + 5$
(b) $x = -8$

8. (a) $k = 1$ (b) $m = 2, n = 1$
(c) $f^2(x) = x$ (d) $f^{-1}(2) = 3$

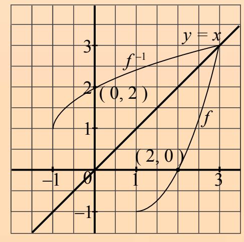
9. (a) (i) Continuous functions (ii) $-4 \leqslant f(x) \leqslant 4$
(b) Does not have an inverse function

10. (a) Condition $x \geqslant 0$. (b) $f^{-1}(x) = x, f^{-1}(x) = x^{\frac{1}{4}}$

11. A graph does not have to cross the line $y = x$ if the graph of a function and its inverse intersect. Both of these graphs might intersect at another line.

12. (a) (i) $f^{-1}(x) = \dfrac{8+5x}{x-1}, x \neq 1$
(ii) $f^{-1}(x) = \dfrac{-3-4x}{x-2}, x \neq 2$
(b) $f = f^{-1}$ if $a = -d$

13. (a) (i) (ii)



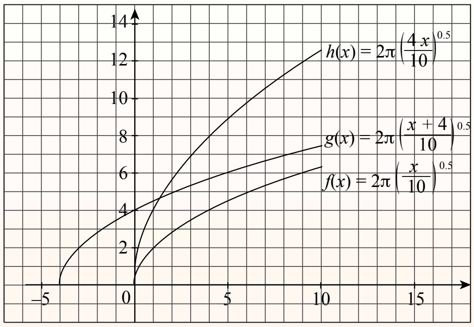
The range of f is $-1 \leqslant f(x) \leqslant 3$ and the domain of f^{-1} is $-1 \leqslant x \leqslant 3$.

(b) Range of f = domain of f^{-1} and domain of f = range of f^{-1}.

(i) Yes
(ii) Yes, any points of (b, a) on the graph of f^{-1} are the reflection points of (a, b) on the graph of f at the line $y = x$.

14. (a) $C = \dfrac{2\sqrt{100 - p}}{25} + 600$
(b) RM600.64

15.



The period of pendulum T depends on the length of the pendulum, l. If the length increases, the period of oscillation of the pendulum also increases.

CHAPTER 2 QUADRATIC FUNCTIONS

Self Practice 2.1

1. (a) –5.606, 1.606 (b) –1.193, 4.193
(c) –7.243, 1.243 (d) 0.634, 2.366
(e) 0.134, 1.866 (f) –0.712, 4.212
2. (a) –1.317, 5.317 (b) –1.366, 0.366
(c) 0.131, 2.535 (d) –0.425, 1.175
(e) –0.449, 4.449 (f) 0.275, 2.725
3. (a) 8 cm, 6 cm (b) 8 cm × 5 cm
4. 3

Self Practice 2.2

1. (a) $x - 8x + 12 = 0$ (b) $x^2 - 3x - 4 = 0$
(c) $x^2 + 11x + 28 = 0$ (d) $5x^2 + 24x - 5 = 0$
2. $p = 2, q = -9$
3. (a) $5x^2 - 30x + 31 = 0$ (b) $x^2 - 10x - 45 = 0$
(c) $5x^2 - 14 = 0$ (d) $15x^2 - 10x - 3 = 0$
4. (a) $x^2 - 5x - 2 = 0$ (b) $2x^2 - 5x - 1 = 0$
(c) $4x^2 - 29x + 1 = 0$ (d) $2x^2 + 29x + 2 = 0$
5. $8x^2 + 36x - 27 = 0$

Self Practice 2.3

1. (a) $-2 < x < 2$ (b) $2 < x < 8$
(c) $-2 \leqslant x \leqslant 6$ (d) $x \leqslant -1$ or $x \geqslant 3$
(e) $-3 < x < 1$ (f) $\dfrac{2}{3} < x < 4$
2. $x \leqslant -2$ or $x \geqslant 8$

Intensive Practice 2.1

1. 0.059, 5.607
2. (a) $x^2 - 12x + 11 = 0$ (b) 12, 11
3. (a) $19x^2 - 4x - 1 = 0$
(b) $7x^2 + 160x + 175 = 0$
(c) $x^2 + 12x + 13 = 0$
4. $k = -14$
5. (a) $r = 1$ (b) $r = -3$
(c) $r = -2$ or $r = \dfrac{1}{16}$
6. $m = 12$; 2, 6
7. 2 and 4; $k = 8$
8. –12, 12
9. $h = 2, k = -5$
10. $c = \dfrac{64 - 9d^2}{4}$
11. (a) $x \leqslant -\dfrac{1}{2}$ or $x \geqslant 1$ (b) $1 \leqslant x \leqslant 4$
(c) $-4 < x < 4$
12. (a) $m = -1, n = 12$ (b) $m = -20, n = 6$
13. $a = 3, b = -10$

Self Practice 2.4

1. (a) 12; two different real roots
(b) 0; two equal real roots
(c) –104; no real roots
(d) 109; two different real roots
(e) 0; two equal real roots
(f) 49; two different real roots

Self Practice 2.5

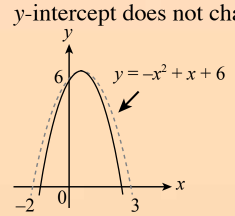
1. (a) $p = -\dfrac{3}{4}$ or $p = 3$ (b) $p > -\dfrac{3}{4}$
(c) $p < \dfrac{3}{4}$
2. $k < -2$ or $k > 6$, $k = -2$ or $k = 6$
3. (a) $h = -4, k = -12$ (b) $c < -16$
4. $k = \dfrac{5}{4}h$
5. 5 : 4

Intensive Practice 2.2

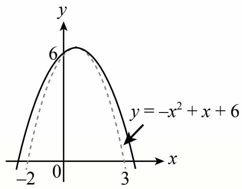
1. (a) Two equal real roots
(b) Two different real roots
(c) No real roots
2. (a) $k = -4$ or $k = 8$ (b) $k = -\dfrac{1}{8}$
3. (a) $r < -3$ or $r > 5$ (b) $r < \dfrac{1}{4}$
4. (a) $p < \dfrac{4}{5}$ (b) $p < -\dfrac{1}{24}$
5. (a) $k = -\dfrac{5}{3}$ or $k = \dfrac{5}{2}$ (b) $x = -3$
6. $m = 2n - 4$
7. (a) $b = 8, c = 12$ (b) –6, –2
8. (a) $c_1 = 4, c_2 = 5$
(b) The equation does not have two real roots

Self Practice 2.6

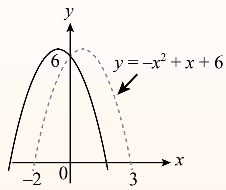
1. (a) (i) The width of the graph decreases, y-intercept does not change.



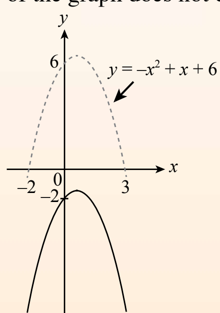
(ii) The width of the graph increases, y-intercept does not change.



(b) The vertex is on the left side of y-axis. All points are changed except for y-intercept. The shape of the graph does not change.

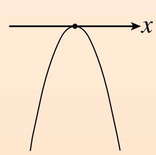


(c) The graph moves 8 units downwards. The shape of the graph does not change.

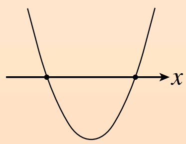


Self Practice 2.7

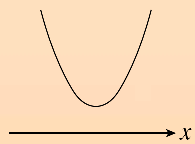
1. (a) The quadratic function has two equal real roots. The graph is a parabola that passes through the maximum point and touches the x-axis at a point.



(b) The quadratic function has two different real roots. The graph is a parabola that passed through the minimum point and intersects the x-axis at two points.



(c) The quadratic function does not have real roots. The graph is a parabola that passes through the minimum point and above the x-axis.



2. (a) $-1, 2$ (b) $1, 5$

3. (a) $q < 7$ (b) $q > -\frac{10}{3}$

4. (a) $r < -\frac{2}{3}$ (b) $r > \frac{4}{3}$

Self Practice 2.8

1. $a = 2, p = 1, q = 5$

2. (a) $f(x) = x^2 - 4x + 3$,
$f(x) = (x - 1)(x - 3)$
(b) $f(x) = -4x^2 + 4x + 8$,
$f(x) = -4(x + 1)(x - 2)$
(c) $f(x) = 2x^2 + 4x - 16$,
$f(x) = 2(x + 4)(x - 2)$

3. The vertex is $(-4, -5)$, $f(x) = -\frac{1}{2}x^2 - 4x - 13$

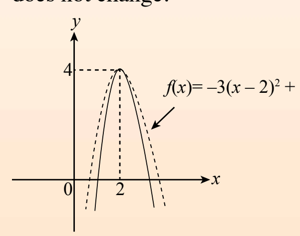
4. (a) $a = -1, h = 2, k = 16$
(b) $f(x) = -x^2 - 4x + 12$
$f(x) = -(x + 6)(x - 2)$

5. (a) $f(x) = \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$
(b) $f(x) = -(x + 1)^2 + 5$
(c) $f(x) = -2\left(x + \frac{1}{4}\right)^2 + \frac{49}{8}$
(d) $f(x) = 3\left(x - \frac{1}{3}\right)^2 - \frac{28}{3}$
(e) $f(x) = -(x - 2)^2 + 16$
(f) $f(x) = 2(x + 1)^2 - 18$

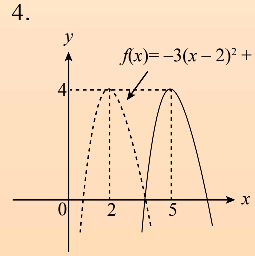
Self Practice 2.9

1. (a) The maximum point is $(2, 4)$ and the equation of the axis of symmetry is $x = 2$.

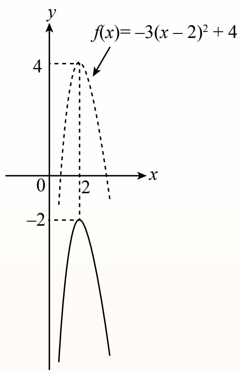
(b) (i) When a changes from –3 to –10, the width of the graph decreases. The axis of symmetry $x = 2$ and the maximum value 4 does not change.



(ii) When h changes from 2 to 5, the graph with the same shape moves horizontally 3 units to the right. The equation of the axis of symmetry becomes $x = 5$ and the maximum value does not change which is 4.



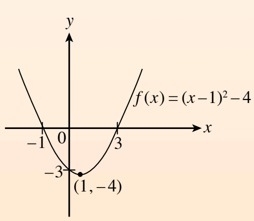
(iii) When k changes from 4 to –2, the graph with the same shape moves vertically 6 units downwards. The maximum value becomes –2 and the axis of symmetry does not change.



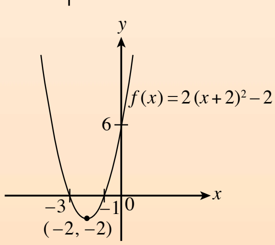
2. (a) $h = 3, k = -3, p = 3$
   (b) $x = 5$ (c) -1
3. (a) The graph moves 6 units to the right and the width of the graph increases. The equation of the axis of symmetry becomes $x = 6$ and the minimum value does not change, which is 0.
   (b) The graph moves 1 unit to the right and 5 units upwards and the width of the graph decreases. The equation of the axis of symmetry becomes $x = 1$ and the minimum value becomes 5.
   (c) The graph moves 1 unit to the left and 4 units downwards and the width of the graph increases. The equation of the axis of symmetry becomes $x = -1$ and the minimum value becomes -4.

Self Practice 2.10

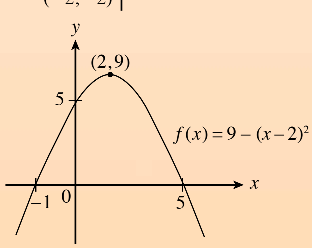
1. (a)



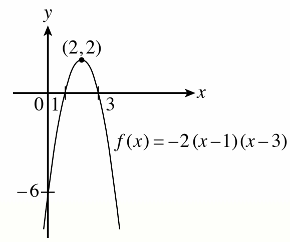
(b)



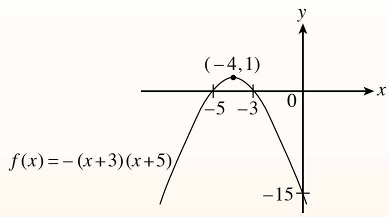
(c)



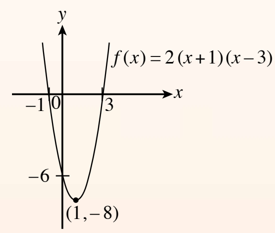
(d)



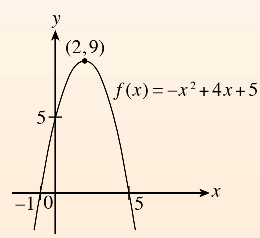
(e)



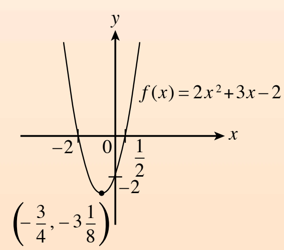
(f)



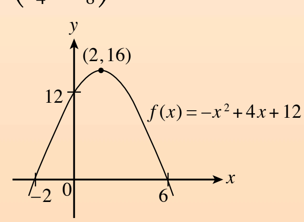
(g)



(h)



(i)

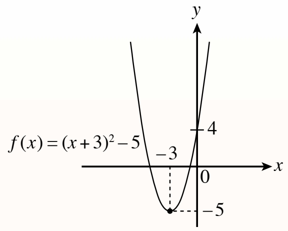


Self Practice 2.11

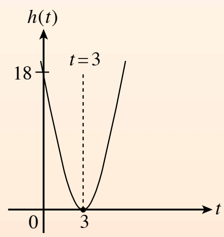
1. (a) 4 m (b) 0.8 second
   (c) 7.2 m (d) $0 < t < 2$
2. (a) 15 m (b) 31.62 m
3. 4 m, 1 m
4. (a) 200 m (b) 50 meter

Intensive Practice 2.3

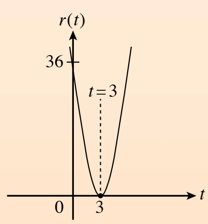
1. (a) $k = -1$ or $k = 4$ (b) $k > -\frac{7}{3}$
2. 5
3. (a) $(x + 3)^2 - 9 + n$ (b) 4
   (c)



4. $-6 < r < -2$, $r = -6$ or $r = -2$
5. (a) The width of the graph decreases. The axis of symmetry and the minimum value do not change.
   (b) The graph with the same shape moves horizontally 3 units to the right. The equation of the axis of symmetry becomes $x = 4$ and the minimum value does not change, which is 2.
   (c) The graph with the same shape moves vertically 3 units upwards. The minimum value becomes 5 and the axis of symmetry does no change, which is $x = 1$.
6. (a)



   (b)



   (c) The graph of function $h(t)$ with the value of $a = 2$ is wider than the graph of $r(t)$ with the value of $a = 4$. Therefore, the bird that is represented by the function $r(t)$ moves at the highest position, which is 36 m above the water level compared to the bird that is represented by the function $h(t)$ with 18 m above the water level.
7. $p = 3$, $q = 7$
8. (a) $b = -1$ (b) $c > 2$ (c) $c = 4$
9. (a) 4 seconds (b) 64 m
10. (a) (i) α (ii) β
    (iii) $-\alpha\beta$ (iv) $\frac{\alpha + \beta}{2}$
    (b) $\frac{\alpha + \beta}{2}$ is the x-coordinate of the maximum point of the graph and $-\alpha\beta$ is the y-intercept of the graph.

MASTERY PRACTICE

1. -0.816, 3.066
2. (a) $x^2 - 8x + 13 = 0$ (b) 8, 13
   (c) Two real and different roots
3. (a) $k = -8, 4$ (b) $k < -8$, $k > 4$
   (c) $k \leqslant -8$, $k \geqslant 4$
4. (a) $p = 2$ (b) $p = -1$
5. $h : k = 7 : 6$; $x = 1$
6. $x < 2$ or $x > 5$, $0 \leqslant x \leqslant 7$; $0 \leqslant x < 2$ or $5 < x \leqslant 7$
7. (a) 3, 7 (b) $p = -5$, $q = -12$
   (c) $x = 5$ (d) $3 < x < 7$
8. (a) $b = -8$, $c = 12$ (b) $(4, -4)$
   (c) $2 < x < 6$ (d) 4
9. 9 km/h
10. 67.229 units
11. (a) 20 units (b) 20 units
12. (a) $y = -\frac{1}{18}(x - 3)^2 + 2.5$
    (b) 9.708 m

CHAPTER 3 SYSTEMS OF EQUATIONS

Self Practice 3.1

1. $3x + 2y + z = 750$
2. (a) Yes, because all three equations have three variables, m, n and p where the power of the variables is 1. The equation has zero n value.
   (b) No, because there are equations with have the variable of power of 2.
   (c) Yes, because all three equations have three variables, a, b and c where the power of the variables is 1.

Self Practice 3.2

1. (a) $x = 1, y = 3, z = 2$ (b) $x = -1, y = 2, z = 3$
2. (a) $x = -1, y = 3, z = -1$
   (b) $x = -\frac{28}{3}, y = 8, z = \frac{16}{3}$

Self Practice 3.3

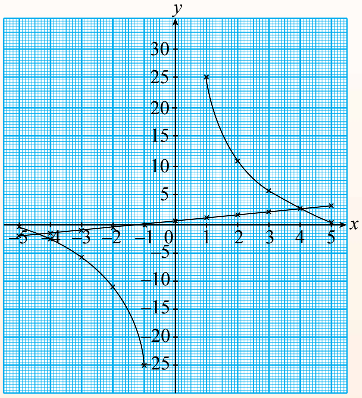
1. P = RM8 000, Q = RM2 000, R = RM14 500
2. Carnations = 80, roses = 50, daisies = 70
3. Pens = 3, pencils = 5, notebooks = 8

Intensive Practice 3.1

1. (a) $x + y + z = 180$, $x - 20 = y + z$, $x - 10 = 3z$; 100°, 50°, 30°
   (b) $x + y + z = 19$, $2x + y + z = 22$
   $x + 2y + z = 25$; 3, 6, 10
2. (a) $x = 2, y = 1, z = 0$ (b) $x = 3, y = 2, z = 1$
   (c) $x = 5, y = -3, z = 1$ (d) $x = \frac{8}{5}, y = -\frac{44}{5}, z = -6$
   (e) $x = -1, y = 3, z = 1$ (f) No solution
3. Butters = 500, chocolates = 750, coconuts = 900
4. Small = 9, medium = 6, large = 3
5. Chickens = 20, rabbits = 10, ducks = 20

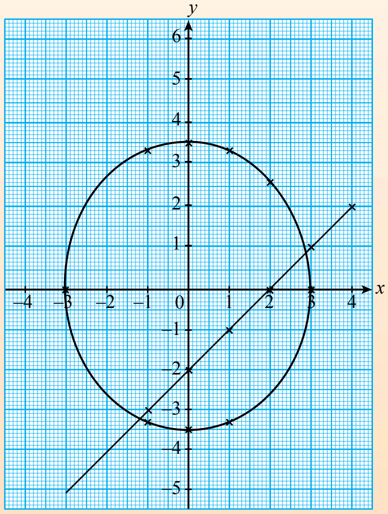
Self Practice 3.4

1. (a) $x = 19, y = 31$ and $x = 2, y = -3$
   (b) $x = -\frac{7}{3}, y = \frac{2}{3}$ and $x = -4, y = 1$
   (c) $x = 3.5811, y = -0.5811$ and $x = 0.4189, y = 2.5811$
   (d) $x = 7, y = -4$ and $x = -\frac{14}{3}, y = 3$
   (e) $x = \frac{13}{4}, y = \frac{5}{8}$ and $x = \frac{7}{2}, y = \frac{1}{2}$
   (f) $x = 3, y = 1$ and $x = -3, y = 7$
2. (a)



$(-4.3, -1.7)$ and $(4.0, 2.5)$

   (b)



$(-1.2, -3.2)$ and $(2.9, 0.9)$

Self Practice 3.5

1. 8 cm, 9 cm
2. $x = 8$ cm, $y = 6$ cm or $x = 6$ cm, $y = 4$ cm

Intensive Practice 3.2

1. (a) $x = 5, y = 3$ and $x = -6, y = -\frac{2}{3}$
   (b) $k = 3.7322, p = 1.5774$ and $k = 0.2678, p = 0.4226$
2. $\left(\frac{6}{5}, \frac{3}{5}\right)$ and $(3, -3)$
3. $h = -2, k = \frac{1}{2}; x = 1, y = -4$
4. $x = 5, y = 7$
5. 35.8519 cm^3 or 36 cm^3
6. $(-1, 0)$ and $\left(-\frac{17}{29}, \frac{4}{29}\right)$
7. $(-1, -2)$ and $\left(\frac{8}{3}, \frac{5}{3}\right)$

MASTERY PRACTICE

1. (a) $x + 2y + 3z = 120, 2x + 3y + 2z = 110$
   $x + 4y + 2z = 180$
   (b) $x + y + z = 30, 10x + 20y + 50z = 2\,060,$
   $x - 3y - 2z = 25$
2. (a) $x = -2, y = 1, z = 3$
   (b) $x = -1, y = 2, z = -4$
3. $x = 15, y = 110, z = 55$
4. $h = 2; x = -\frac{1}{7}, y = \frac{2}{7}$
5. RM 13 166.67, RM 6 666.67, RM166.66
6. 8 m, 15 m and 17 m
7. Yes, the line crossed the curve at another point, which is $x = \frac{5}{2}, y = \frac{9}{2}$
8. 48 cm^2
9. 4.5 m, 5.5 m
10. $x = \frac{2}{3}, y = 12$ and $x = \frac{1}{3}, y = 24$
11. Diameter = 7 m, radius = $\frac{7}{2}$ m; Diameter = $\frac{28}{9}$ m, radius = $\frac{14}{9}$ m

CHAPTER 4 INDICES, SURDS AND LOGARITHMS

Self Practice 4.1

1. (a) 5^{5x} (b) $\frac{1}{7^5} - \frac{1}{7^3}$
   (c) $9^a(9^{-5} + 9^2)$ (d) c^7d^8
   (e) x^6y^{11} (f) $\frac{xy^3}{49^5}$
   (g) $27x^2y$ (h) $p^{10}q^3$
   (i) p^7q^{20} (j) $\frac{x^5y^5}{7^{10}}$
   (k) $\frac{5y^6}{x^{10}}$ (l) $\frac{a^4b^2}{6}$
2. (a) $\frac{2}{a^{\frac{1}{6}}}$ (b) $4a^{\frac{18}{5}}$
   (c) $\frac{1}{a^{\frac{17}{20}}}$ (d) $\frac{1}{a} + \frac{3}{a^3} - \frac{3}{a^4}$

Self Practice 4.2

1. (a) $x = -11$ (b) $x = -2$
   (c) $x = -3$
2. (a) 10 cm (b) 3.4868 cm

Intensive Practice 4.1

1. (a) $\frac{y^3z^2}{x}$ (b) $\frac{xz^3}{y}$
(c) $4x^{12}y^{16}$ (d) e^5f^4
(e) $\frac{7y^9}{x^5}$ (f) $\frac{x^3y^5}{7^8}$
2. $x = 7$
3. $x = 4$
4. $m = 1$
5. $-\frac{1}{2} \rightarrow -2 \rightarrow -\frac{2}{3} \rightarrow \frac{3}{10} \rightarrow 2 \rightarrow 3 \rightarrow -1 \rightarrow -3 \rightarrow -3$
6. (a) 5 904 900 (b) 5 minutes
7. 79 570 057
8. RM51 874.85

Self Practice 4.3

1. (a) $\frac{26}{33}$ (b) $3\frac{19}{33}$
(c) $\frac{115}{333}$ (d) $13\frac{21}{37}$
2. (a) Surd because it is a non-recurring decimal.
(b) Surd because it is a non-recurring decimal.
(c) Not surd because it is a recurring decimal.
(d) Surd because it is a non-recurring decimal.

Self Practice 4.4

1. (a) $\sqrt{6}$ (b) $\sqrt{15}$
(c) $\sqrt{9}$ (d) $\sqrt{30}$
(e) $\sqrt{\frac{8}{3}}$ (f) $\sqrt{6}$
(g) $\sqrt{4}$ (h) $\sqrt{10}$

Self Practice 4.5

1. $\sqrt{260} = 2\sqrt{65}$, $(\sqrt{16}\sqrt{36})^2 = 576$,
$\frac{4\sqrt{8}}{2\sqrt{4}} = 2\sqrt{2}$, $\frac{\sqrt{75}}{\sqrt{3}} = 5$, $\frac{30\sqrt{27}}{6\sqrt{3}} = 15$
$(\sqrt{81})^2 = 81$
2. (a) $2\sqrt{3}$ (b) $3\sqrt{3}$
(c) $2\sqrt{7}$ (d) $4\sqrt{2}$
(e) $3\sqrt{5}$ (f) $4\sqrt{3}$
(g) $3\sqrt{6}$ (h) $6\sqrt{3}$

Self Practice 4.6

1. (a) $8\sqrt{5}$ (b) $12\sqrt{5}$
(c) $2\sqrt{7}$ (d) -12
(e) $4\sqrt{5} + 25$ (f) $3\sqrt{7} - 35$
(g) $87 + 35\sqrt{3}$ (h) $20\sqrt{7} - 154$
(i) -133
2. (a) Not similar surds (b) Similar surds
(c) Similar surds (d) Not similar surds
(e) Similar surds

Self Practice 4.7

1. (a) $\frac{2\sqrt{5}}{5}$ (b) $\frac{7\sqrt{2}}{2}$
(c) $\frac{\sqrt{10}}{5}$ (d) $\frac{1}{4}$
(e) $\frac{\sqrt{3}}{6} + \frac{1}{2}$ (f) $\frac{15 + 3\sqrt{5} + 5\sqrt{2} + \sqrt{10}}{20}$
(g) $\frac{16 + \sqrt{3}}{23}$
(h) $\frac{65 + 16\sqrt{2} - 11\sqrt{3}}{46}$
(i) $\frac{45\sqrt{3} - 33\sqrt{5} - 20\sqrt{3}}{55}$

Self Practice 4.8

1. $\sqrt{39}$ cm
2. (a) $\frac{17}{2}$ cm^2 (b) $\sqrt{66}$ cm
3. $13 + 4\sqrt{3}$
4. (a) $x = -2$ (b) $\frac{5}{8}$
(c) $\frac{1}{4}$

Intensive Practice 4.2

1. (a) $\sqrt{55}$ (b) $\sqrt{70}$
(c) $\sqrt{\frac{3}{2}}$ (d) $\sqrt{6}$
2. (a) $2\sqrt{6}$ (b) $9\sqrt{2}$
(c) $3\sqrt{2}$ (d) $\frac{4}{3}\sqrt{4}$
3. (a) $8\sqrt{10}$ (b) $5\sqrt{11}$
(c) $11\sqrt{13}$ (d) $8\sqrt{5}$
(e) $9\sqrt{3} - 6\sqrt{2}$ (f) $3\sqrt{2} + 3\sqrt{3}$
(g) $105\sqrt{3}$ (h) $24\sqrt{30}$
(i) $-\sqrt{3}$ (j) $3\sqrt{7} + 49$
(k) $7\sqrt{5} - 25$ (l) $114 + 24\sqrt{7}$
(m) $-154 - 20\sqrt{7}$ (n) $146 - 50\sqrt{5}$
(o) 4 (p) $\frac{1}{3}$
(q) $\sqrt{2}$ (r) 6
4. (a) $5\sqrt{5} + 7\sqrt{3} - 7\sqrt{7}$
(b) $3\sqrt{5} + 5\sqrt{3} + 18\sqrt{2}$
(c) $13\sqrt{5} + 21\sqrt{3} - 14\sqrt{7}$
(d) $11\sqrt{5} + 17\sqrt{3} - 7\sqrt{7} + 36\sqrt{2}$
5. (a) $\frac{2\sqrt{5}}{5}$ (b) $3 + \sqrt{5}$
(c) $-\frac{(1 + \sqrt{5})}{3}$ (d) $\sqrt{3} + \frac{\sqrt{2}}{2}$
(e) $\frac{17 + 7\sqrt{5}}{4}$ (f) $\frac{-5 + \sqrt{21}}{2}$
6. (a) -1 (b) $\frac{3\sqrt{7} - \sqrt{2}}{5}$

(c) $\frac{12+\sqrt{3}}{13}$

7. $(2\sqrt{5}-\sqrt{2})$ cm

8. (a) $\frac{1+5\sqrt{2}}{7}$ (b) $\frac{7+7\sqrt{2}}{2}$ cm^2

Self Practice 4.9

1. (a) $\log_3 81 = 4$ (b) $\log_2 128 = 7$
(c) $\log_5 125 = 3$ (d) $\log_6 216 = 3$

2. (a) $10^4 = 10\ 000$ (b) $10^{-4} = 0.0001$
(c) $2^7 = 128$ (d) $4^3 = 64$

3. (a) 0.9542 (b) 1.9956
(c) –0.2375 (d) 6
(e) 4 (f) 4
(g) 5

4. (a) $x = 32$ (b) $x = 512$
(c) $x = 256$

5. (a) 138.7714 (b) 24.6774
(c) 5568.0099 (d) 0.0004052
(e) –0.0027064 (f) 0.000027829

Self Practice 4.10

1. (a) 0.115 (b) 1.712
(c) 2.366 (d) –0.712

2. (a) 3 (b) 2
(c) 2

Self Practice 4.11

1. (a) $\log_2 xy^2$ (b) $\log_b \left(\frac{x}{y^3}\right)$
(c) $\log_2 xy^3$ (d) $\log_4 \left(\frac{16\sqrt{x}}{y^3}\right)$
(e) $\log_3 m^3n^2$

2. (a) $1+q$ (b) $2p+q$
(c) $\frac{1}{2}(p+q)$

Self Practice 4.12

1. (a) 2.8137 (b) 0.1550
(c) 1.7959 (d) –0.1475

2. (a) 2.7833 (b) 2.6309
(c) 1.9820

3. (a) $\frac{2}{t}$ (b) $\frac{3t}{2}$
(c) $\frac{2+t}{t}$ (d) $\frac{2-2t}{t}$

4. (a) $\frac{2a+3b}{2}$ (b) $\frac{a-2b}{3}$
(c) $\frac{3+b}{a+b}$

Self Practice 4.13

1. (a) 1.677 (b) 2.399
(c) 1.011

2. (a) 653803.075 (b) –0.712
(c) 1.982 (d) 18.866
(e) 1.792 (f) 6.389, –8.389

3. 2 years

4. 11 years

5. 3 years

6. 5.543 km

Intensive Practice 4.3

1. $\log_5 1 = 0$, $\log_7 75 = 2.219$

2. $2x + y - 3$

3. 2

4. $\frac{4}{3}$

6. $2p - m - 1$

7. $\log_2\left(2+\frac{1}{x}\right)$

8. $y = 2x$

9. $\frac{1}{2}(3+x-y)$

10. (a) 10^{-12} Watt (b) 31 : 25
(c) 140 decibels

11. (a) 2 500 000 (b) 3 729 561
(c) Year of 2095

Self Practice 4.14

1. 3 weeks and 2 days

2. (a) 32 amp
(b) (i) 8 amp (ii) 2 amp
(c) 3 seconds

Intensive Practice 4.4

1. (a) RM1 538.62 (b) 2.1156 years

2. (a) 50 gram (b) 13219.2810 years

3. (b) 6.93 hours

MASTERY PRACTICE

1. $x = 0.4194$

2. $n = 2$

3. $\frac{\sqrt{35}+\sqrt{21}}{2}$

4. $t = 0$

5. $\frac{8}{12+8\sqrt{2}}$

6. (a) 59.05°C (b) 2.12 seconds

7. 9 years

8. $\frac{3}{2s}+\frac{2}{t}$

9. $x = \frac{5}{2}$, $y = 2$

10. 21.85 years

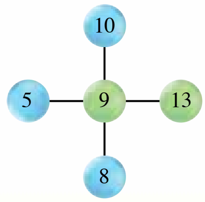
CHAPTER 5 PROGRESSIONS

Self Practice 5.1

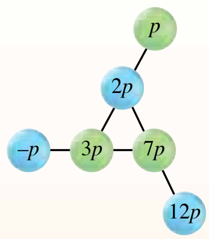
1. (a) 14, add 14 to the previous terms.
(b) $3\sqrt{3}$, add $3\sqrt{3}$ to the previous terms.
(c) $(p-q)$, add $(p-q)$ to the previous terms.
(d) $\log_a 2^3$, add $\log_a 2^3$ to the previous terms.

2. (a) Arithmetic progression
(b) Not an arithmetic progression
(c) Not an arithmetic progression
(d) Arithmetic progression

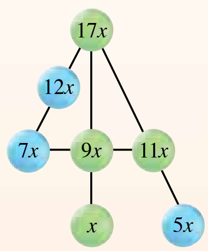
3. (a)



(b)



(c)



4. Arithmetic progression.

Self Practice 5.2

1. $-23 \rightarrow -12 \rightarrow -3 \rightarrow -0.4 \rightarrow 25 \rightarrow -14$

2. (a) 37 years (b) RM1 500

Self Practice 5.3

1. (a) 1 000 (b) $165\frac{3}{5}$

2. Horizontal
(a) 72 (b) 28 700
(c) 300
Vertical
(c) 31 570 (d) 22
(e) 30 100

3. 451 units

4. (a) 14 panels, the remaining wood pieces are 4
(b) White, 27 wood pieces

Self Practice 5.4

1. (a) 21 days (b) 20 books

2. (a) 2 cm (b) 2 cm

Intensive Practice 5.1

1. (a) An arithmetic progression
(b) Not an arithmetic progression

2. (a) 12 (b) -9

3. (a) 12 (b) 14

4. (a) 425 (b) $4n[3n - 13]$
(c) 1 225

5. (a) -3 (b) 29
(c) 85

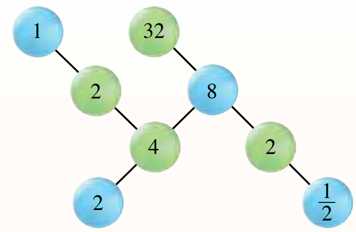
6. (a) -1 (b) $-\frac{23}{2}$

7. Company B, RM2 400

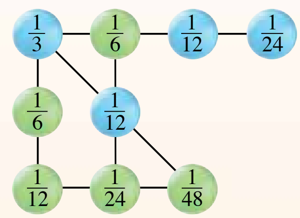
Self Practice 5.5

1. (a) Geometric progression
(b) Geometric progression
(c) Not a geometric progression

2. (a)



(b)



3. $x = 3$
The first three terms: $1, 4, 16$; $r = 4$

Self Practice 5.6

1. $\frac{243}{4} \rightarrow 3 \rightarrow 7 \rightarrow \frac{4}{5} \rightarrow \frac{1}{768} \rightarrow \frac{625}{12} \rightarrow 0.01$

2. 23rd bounce

Self Practice 5.7

1. (a) 81.9 (b) $\frac{p[p^{22} - 1]}{p^2 - 1}$
(c) 3 587 226.5

2. 4

3. (b) 1 365

Self Practice 5.8

1. (a) 2 250 (b) RM30 000
(c) 2 240 (d) 53

Self Practice 5.9

1. (a) 400 cm (b) 8 m

2. (a) The sequences of the perimeter: $\pi j, \pi j(1.4), \pi j(1.4)^2, \ldots$
(b) 24.28 m

Intensive Practice 5.2

1. (a) $n = 8, S_n = 1\,640$ (b) $n = 7, S_n = \log x^{-127}$
(c) $n = 9, S_n = \frac{6}{11}$ (d) $n = 7, S_n = 5\frac{61}{64}$

2. 9

3. (a) $-\frac{1}{2}, 3$ (b) 512

4. (a) $r = \frac{1}{2}, a = 144$ cm^2 (b) 71.72 cm^2

5. (a) $\frac{1}{3}$ (b) 351 cm

6. $r = \frac{1}{2}, T_2 = 7.25$ kg

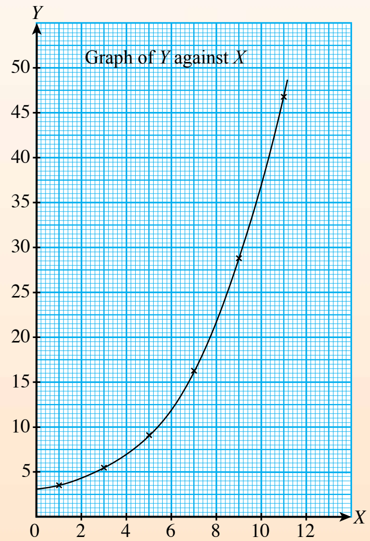
MASTERY PRACTICE

1. (a) 13 (b) −18
2. 3
3. (a) 4 cm^3 (b) 324 cm^3
4. (a) $a = 120, r = \frac{1}{2}$ (b) 240
5. (a) 56 chairs (b) 572 chairs
6. (a) Savings of RM30 000 can be obtained.
   (b) The savings does not reach RM30 000.
7. (a) 3 (b) RM5 460

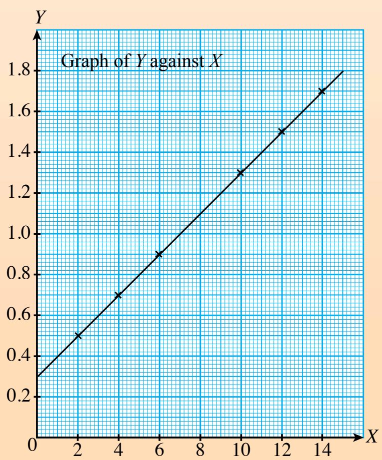
CHAPTER 6 LINEAR LAW

Self Practice 6.1

1. The graph of linear relation is Diagram 1(b). The graph in Diagram 1(a) represents a non-linear relation because the shape of the graph obtained is a curve while the graph in Diagram 1(b) represents a linear relation because a straight line is obtained.
2. (a)



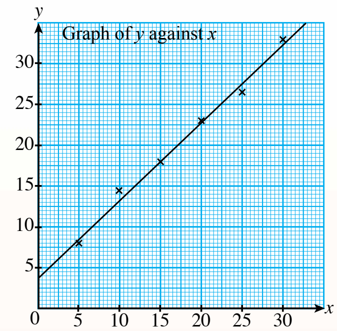
(b)



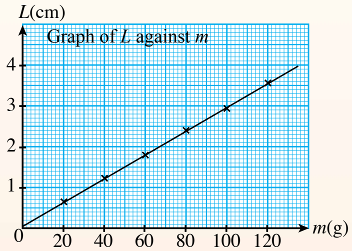
The graph (b) which is a straight line is a graph of linear relation.

Self Practice 6.2

1.

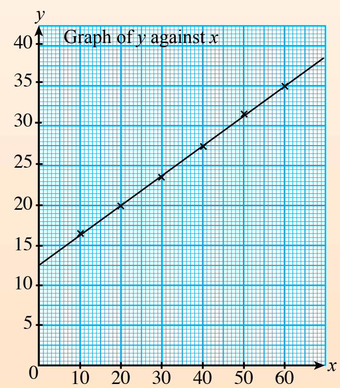


2.



Self Practice 6.3

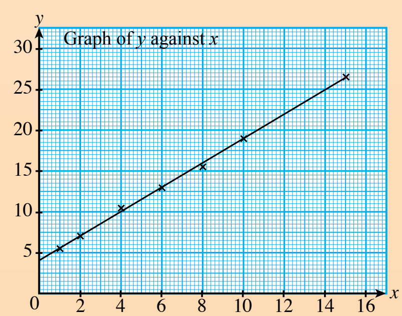
1. $t = \frac{29}{16}x + \frac{1}{50}$
2. (a)



(b) y-intercept = 12.5, gradient = 0.375
(c) $y = 0.375x + 12.5$

Self Practice 6.4

1. (a)

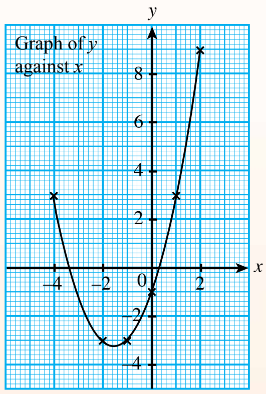


(b) (i) y-intercept = 4.0
(ii) $y = 22$
(iii) Gradient = $\frac{3}{2}$
(iv) $x = 7.4$

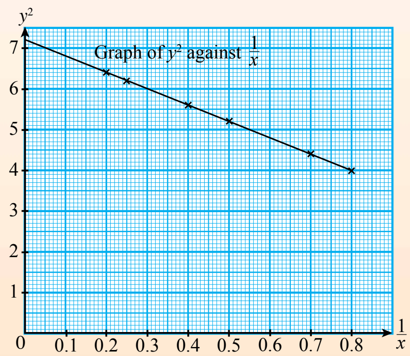
(c) $y = \frac{3}{2}x + 4$; $y = 46$

Intensive Practice 6.1

1. (a)

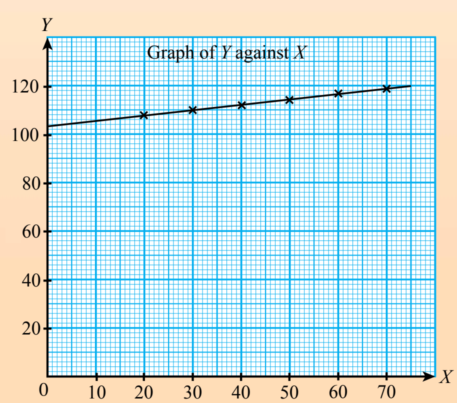


(b)



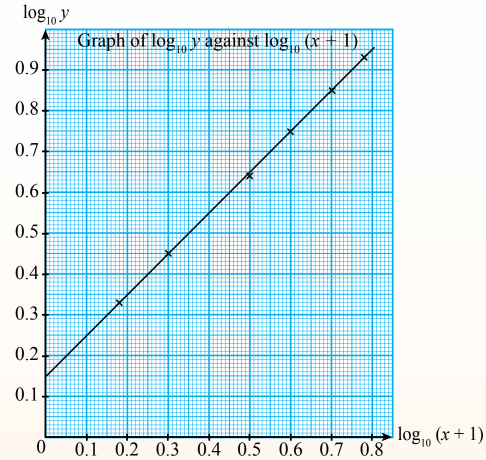
The graph (a) is a non-linear graph while graph (b) is a linear graph. The shape of graph (a) is a curve while the shape of graph (b) is a straight line.

2.



$Y = \frac{11}{50}X + \frac{518}{5}$

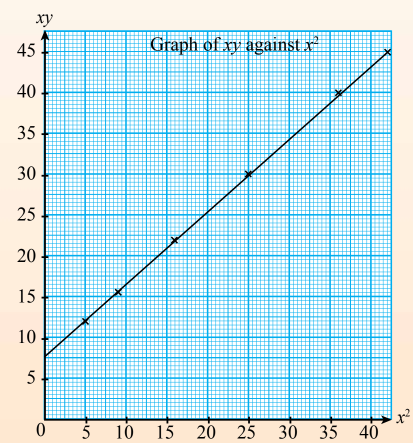
3. (a)



(b) (i) $m = 1$
(ii) 0.15
(iii) 1.512

(c) (i) 4.9442
(ii) 0.0619

4. (a)



(b) (i) $m = \frac{33}{37}$
(ii) 7.5
(iii) 10
(iv) 5.2

(c) $x = 10.18$

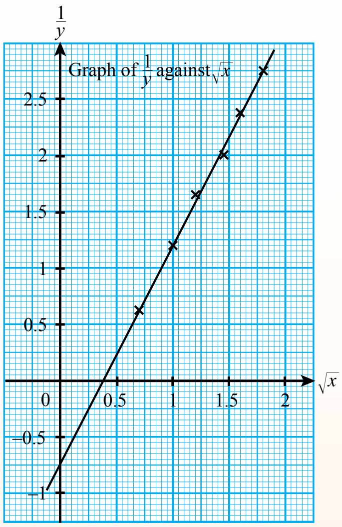
Self Practice 6.5

1. (a) $Y = \frac{y}{x^2}, X = \frac{1}{x^2}, m = -q, c = p$

(b) $Y = \frac{y}{x}, X = x, m = h, c = 1$

(c) $Y = yx^2, X = x^2, m = q, c = p$

2. (a)



(b) (i) $q = -0.75$ (ii) $p = \frac{31}{16}$

(iii) $y = \frac{5}{7}$

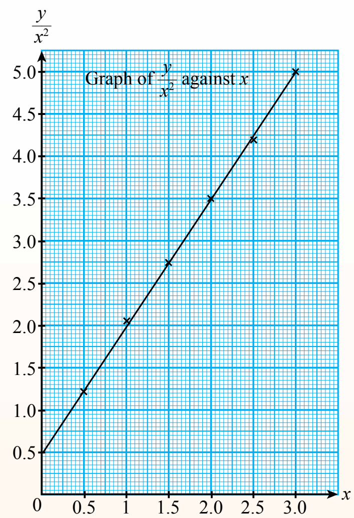
Intensive Practice 6.2

1.

	Non-linear	Linear	*Y*-axis	*X*-axis	Gradient, *m*	*Y*-intercept, *c*
(a)	$y = 5x^2 + 3x$	$\frac{y}{x^2} = 5 + \frac{3}{x}$	$\frac{y}{x^2}$	$\frac{1}{x}$	3	5
(b)	$y = p\sqrt{x} + \frac{q}{\sqrt{x}}$	$y\sqrt{x} = px + q$	$y\sqrt{x}$	x	p	q
(c)	$y = ax^b$	$\log_{10} y = \log_{10} a + b \log_{10} x$	$\log_{10} y$	$\log_{10} x$	b	$\log_{10} a$
(d)	$x = mxy + ny$	$\frac{x}{y} = mx + n$	$\frac{x}{y}$	x	m	n
(e)	$yp^x = q$	$\log_{10} y = -\log_{10} px + \log_{10} q$	$\log_{10} y$	x	$-\log_{10} p$	$\log_{10} q$
(f)	$y(b - x) = ax$	$\frac{x}{y} = -\frac{x}{a} + \frac{b}{a}$	$\frac{x}{y}$	x	$-\frac{1}{a}$	$\frac{b}{a}$

2. (a) $\frac{y}{x^2} = ax + b$

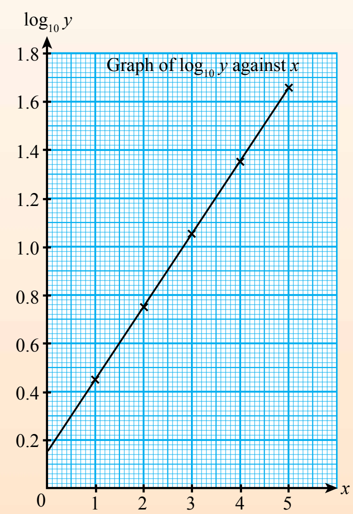
(b)



(c) $a = 1.504$
$b = 0.5$

3. (a) $\log_{10} y = b \log_{10} a + x \log_{10} a$

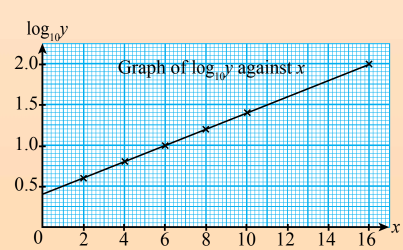
(b)



(c) $a = 2.0106$
$b = 0.5275$

Self Practice 6.6

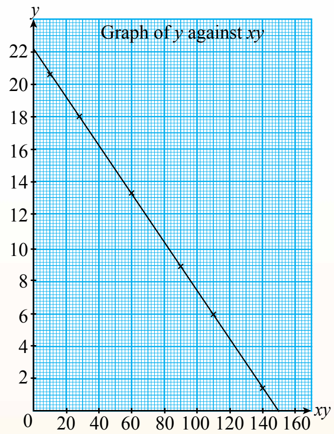
1. (a)



(b) (i) $p = 2.51189$
(ii) $q = 1.25893$

(c) $y = 7.94328$

2. (a)



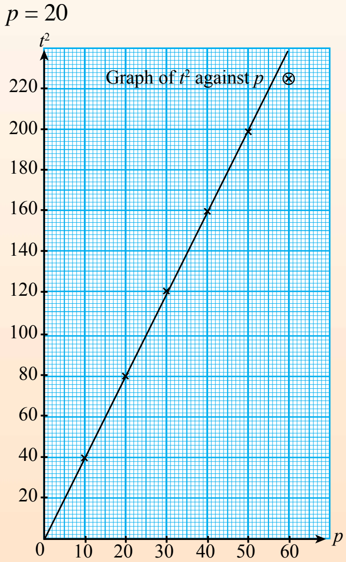
(b) $a = -150.314868$
$b = -6.77094$

(c) Gradient = 0.0066899, Y-intercept = 0.0452991

Intensive Practice 6.3

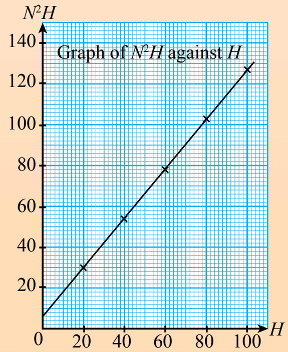
1. (a) $p = 10$
(b) $p = 20$

2. (a)



(b) $t = 15.5$
(c) $k = 2$

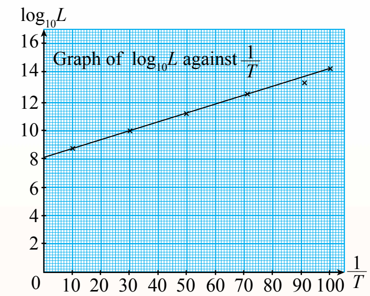
3. (a)



(b) $a = 2.425$
$b = 12$

(c) $N = 1.3416$
(d) 20 workers

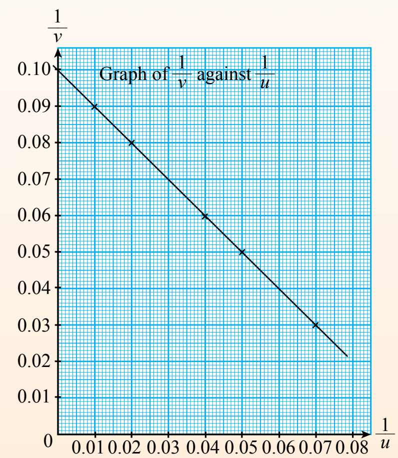
4. (a)



(b) (i) $A = 1.585 \times 10^8$
(ii) $b = 0.1258$

(c) 221.7°C

5. (a)



(b) (i) $v = \dfrac{-10u}{10 - u}$ (ii) 10

MASTERY PRACTICE

1. (a) $yx^2 = 3x^3 + 4,\ \dfrac{y}{x} = 3 + \dfrac{4}{x^3}$

(b) $\dfrac{y}{x^2} = px + q,\ \dfrac{y}{x^3} = p + \dfrac{q}{x}$

(c) $xy = p + \dfrac{q}{p}x^2,\ \dfrac{y}{x} = \dfrac{p}{x^2} + \dfrac{q}{p}$

(d) $\log_{10} y = \log_{10} p + \sqrt{x}\log_{10} k$

(e) $\log_{10} y = \log_{10} p + (x - 1)\log_{10} k$

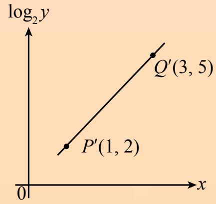
(f) $\log_{10} y = x^2 \log_{10} k - \log_{10} p$

2. (a) $\dfrac{y}{x} = px + q$ (b) $p = -0.25, q = 5.25$

3. $p = 100, q = 100$

4. $k = -1, h = \dfrac{2}{3}$

5. (a)



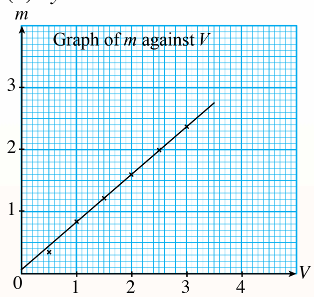
$P'(1, 2), Q'(3, 5)$

(b) $a = 1.414, b = 2.828$

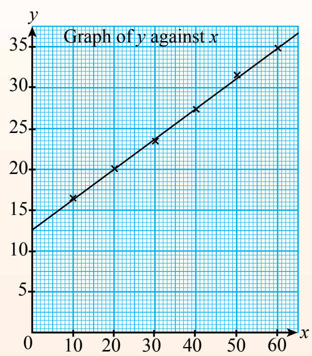
6. (a) $y = \dfrac{8x + 3}{x^2}$

(b) $y = 0.8850$

7.

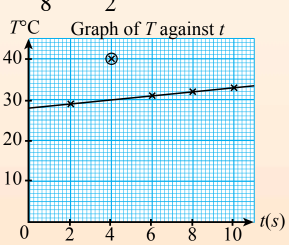


8. (a)



(b) $y = \dfrac{3}{8}x + \dfrac{25}{2}$

9. (a)

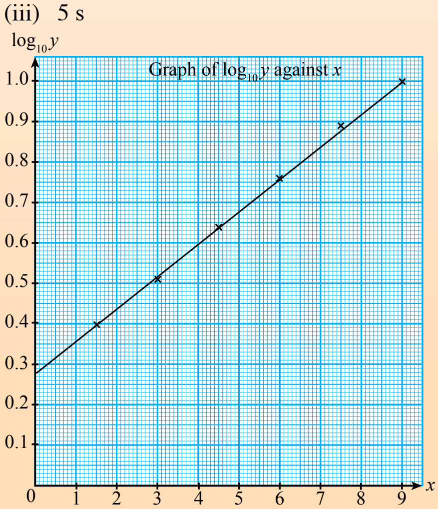


(b) 30.0

(c) (i) 28°C (ii) 32.5°C

(iii) 5 s

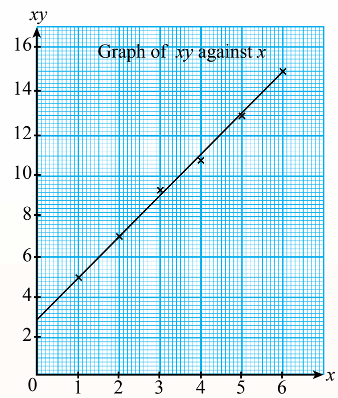
10. (a)



(b) (i) $s = 1.90546$ (ii) $t = 1.20226$

(iii) $x = 4$

11. (a)



(b) (i) $p = 4$ (ii) $q = 6$

(iii) $y = 2.8571$

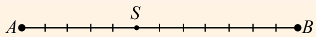
(c) $x = 0.0625$

CHAPTER 7 COORDINATE GEOMETRY

Self Practice 7.1

1. (a) Point P divides line segment AB in the ratio 1 : 2.
Point Q divides line segment AB in the ratio 1 : 1.
Point R divides line segment AB in the ratio 11 : 1.

(b)



2. (a) $m = 2, n = 5$

(b) P divides rope AB in the ratio 2 : 5.

(c) $P(6, 0)$

Self Practice 7.2

1. (a) $P(-3, 4)$ (b) $P(-2, 1)$

(c) $P(3, -1)$

2. $p = -2t$

3. (a) $C(4,4)$ (b) $D(2,1)$

4. (a) $1 : 2, k = -2$ (b) $1 : 1, k = 5$

(c) $1 : 4, k = 7$ (d) $2 : 3, k = 2$

Self Practice 7.3

1. (28, 32)

2. (–1, 4), (2, 3)

3. (a) 2 : 1 (b) 5 units

Intensive Practice 7.1

1. $R(6, 4)$

2. (a) $Q(11, -2)$ (b) $\left(\dfrac{15}{2}, \dfrac{3}{2}\right)$

3. $h = 7, k = 1$

4. $e = 10f$

5. (a) $U(5, -4)$ (b) $\left(8, -\dfrac{3}{2}\right)$

(c) 3 : 1 (d) 5 units

6. (a) 1 : 3 (b) –1

7. $\left(\dfrac{17}{2}, 4\right)$

Self Practice 7.4

1. (a) Parallel (b) Parallel
   (c) Perpendicular (d) Perpendicular
2. (a) $-\frac{1}{6}$ (b) 2
3. (a) 3 (b) 6
4. 8

Self Practice 7.5

1. $3y - 2x = 20$
2. (a) $(5, 5)$ (b) 3.606 units

Intensive Practice 7.2

1. (a) Parallel (b) Perpendicular
2. 3
3. (a) $3y + 2x = 23$ (b) $2y - 3x = 11$
   $S(1, 7)$
4. (a) -9 (b) 17
5. $h = -2$
6. (a) $2y + x = 10, y = 2x$ (b) $C(2, 4)$, 4.472 units
7. (a) AB is $3y - x = 5$
   DE is $y + 3x = 15$
   (b) $E(4,3), B(7,4)$
8. (a) AB is parallel to CD, AB is perpendicular to AD, CD is perpendicular to AD.
   (b) $2y = x + 9$
   (c) $y + 2x - 22 = 0$
9. (a) (i) $3y + 2x = 19$ (ii) $B(8, 1)$
   (b) (i) $D(2, 5)$
11. $2y + x = 17$

Self Practice 7.6

1. (a) 24 units^2 (b) 12 units^2
   (c) $28\frac{1}{2}$ units^2
2. $(5, 0), \left(-\frac{5}{3}, 0\right)$
4. $p = 1$
5. (a) $-3, \frac{11}{3}$ (b) 1, 21
   (c) $-5, 7$ (d) $-7, 3$

Self Practice 7.7

1. (a) 52 units^2 (b) $88\frac{1}{2}$ units^2
   (c) 19 units^2 (d) $27\frac{1}{2}$ units^2
2. $k = -4$

Self Practice 7.8

1. $46\frac{1}{2}$ units^2
2. 30 units^2

Self Practice 7.9

1. (a) $C(7, 8), M(2, 4)$ (b) 1 : 4
2. (a) $k = 2$ (b) $P(3,2)$
3. (a) $13\frac{1}{2}$ units^2 (b) $k = 1$
   (c) $E(7, -4)$ (d) 27 units^2

Intensive Practice 7.3

1. (a) $D(-2, 10), E(-1, 4)$ (b) 50 units^2
2. (a) $h = -2, k = -1$ (b) 20 units^2
3. (a) 0
   (b) Point A, B and C are calinear.
4. $47\frac{1}{2}$ units^2
5. 5, 37
6. 1, 5
7. (a) 20 (b) 14, 26
8. (a) $k = 7$
   (b) (i) $H(3, 11)$ (ii) 1 : 2
9. (a) $m = 2$ (b) 17 units^2
10. (a) 1.1402 km (b) 0.645 km^2

Self Practice 7.10

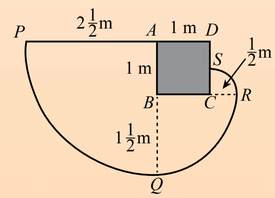
1. (a) $x^2 + y^2 - 9 = 0$
   (b) $x^2 + y^2 - 4x - 6y + 4 = 0$
   (c) $x^2 + y^2 + 8x - 10y + 32 = 0$
   (d) $x^2 + y^2 + 2x + 12y + 28 = 0$
2. $x^2 + y^2 + 4x - 2y - 20 = 0$
3. (a) $x^2 + y^2 + 8x = 0$
   (b) $4x^2 + 4y^2 + 29x + 5y + 26 = 0$
   (c) $5x^2 + 5y^2 + 36x - 56y + 164 = 0$
   (d) $x^2 + y^2 - 10x + 4y + 21 = 0$
4. $5x^2 + 5y^2 + 50x - 6y - 118 = 0$
5. $x^2 + y^2 + 12x = 0$
6. $15x^2 + 15y^2 + 4x - 4 = 0$
7. (a) $x + 2y - 3 = 0$ (b) $5x - 9y + 7 = 0$
   (c) $8x + 10y - 87 = 0$

Self Practice 7.11

1. $x^2 + y^2 - 6x - 8y + 9 = 0$
2. (a) $x - y - 4 = 0$ (c) $(7, 3), (12, 8)$

Intensive Practice 7.4

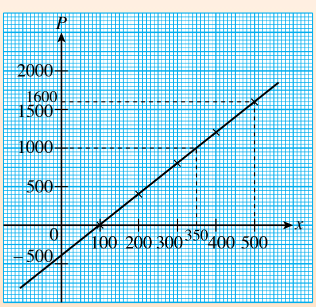
1. (a) $3x^2 + 3y^2 + 12x - 68y + 364 = 0$
   (b) $\left(0, \frac{26}{3}\right), (0, 14)$
2. $x^2 + y^2 - 8x - 10y + 16 = 0$
3. (a) $x^2 + y^2 - 12x - 10y + 36 = 0$
   (b) 2, 10
4. $y^2 = 4x$
5. (a) $\alpha^2 + \beta^2 = 81$ (b) $4x^2 + y^2 = 36$
6.



The locus consists of curves of a quadrant of 3 circles:
(i) APQ that is a quadrant with centre A and radius of $2\frac{1}{2}$ m
(ii) BQR that is a quadrant with centre B and radius of $1\frac{1}{2}$ m
(iii) CRS that is a quadrant with centre C and radius of $\frac{1}{2}$ m

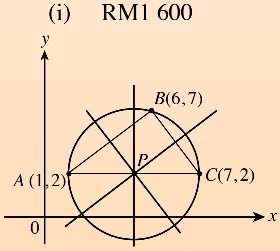
MASTERY PRACTICE

1. (a) $h = -3, k = 5$ (b) $\frac{2}{5}$
   (c) $2y + 5x = 16$
2. (a) $P(2, 2)$ (b) $y = x$
3. $\frac{1}{2}, 1$
4. $(0, 6), (0, -3)$
5. $2x^2 + 2y^2 + 19x + 35 = 0$
6. $(3, 3)$
7. (a) $C(4, -3)$ (b) $D(8, 7)$
   (c) (i) $k = -\frac{1}{2}$
8. (a) $P(3, 1)$
   (b) QR: $y + 3x = 40$ SR: $3y - x = 10$
   (c) $Q(12, 4), S(5, 5)$ (d) 25 units^2
9. (a) 30 units^2
   (b) $\frac{9k - 4h - 1}{2}, 37 - 3k - 2h$ (c) $P(6, 5)$
   (d) $y = x - 1$
   (e) (i) $Q(8, 7)$ (ii) 1 : 1
10. (a) $R(-3, 6), S\left(0, \frac{15}{2}\right), T\left(\frac{15}{8}, \frac{15}{4}\right)$
    (b) $18\frac{9}{32}$ units^2
11. (a) $h = 1, k = 4$ (b) $y + 2x = 10$
    (c) $y = -2x + 8, y = -2x - 8$
12. (a) $y + 5x + 9 = 0$
    (b) $P(-3, 6), D(7, 8), C(13, 4)$
    (c) 78 units^2
13. (a) $E(3, 1)$ (b) Square
    $B(6, -3)$
14. (a) $P = 4x - 400$
    (b)



    (i) RM1 600 (ii) 350 copies

15.



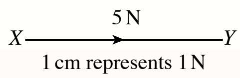
CHAPTER 8 VECTORS

Self Practice 8.1

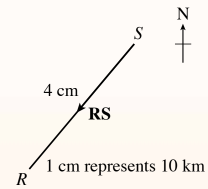
1. (a) Scalar quantity because the quantity only consists of magnitude.
   (b) Vector quantity because the quantity consists of magnitude and direction.
   (c) Scalar quantity because the quantity only consists of magnitude.
   (d) Scalar quantity because the quantity only consists of magnitude.
   (e) Vector quantity because the quantity consists of magnitude and direction.

Self Practice 8.2

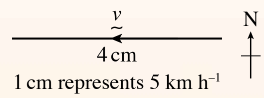
1. (a)



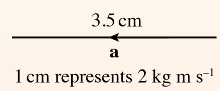
   (b)



   (c)



   (d)



2. $\sqrt{20}$ N, 026.57°
3. 117.15 km
4. $\overrightarrow{MN} = \overrightarrow{CD}, \overrightarrow{EF} = \overrightarrow{KL}, \overrightarrow{GH} = \overrightarrow{AB}, a = d, c = f, b = e$
5. (a) (i) $\overrightarrow{ED}$ (ii) $\overrightarrow{FE}$
   (iii) $\overrightarrow{AF}$
   (b) (i) $\overrightarrow{DC}$ (ii) $\overrightarrow{CB}$
   (iii) $\overrightarrow{BA}$

Self Practice 8.3

1. $\overrightarrow{PQ} = \frac{1}{2}\underset{\sim}{a}, \underset{\sim}{x} = -\frac{3}{2}\underset{\sim}{a}, y = -\frac{7}{4}\underset{\sim}{a}, \overrightarrow{RS} = \frac{5}{4}\underset{\sim}{a}$

Self Practice 8.4

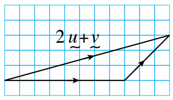
1. $\overrightarrow{AB} = \frac{1}{4}\overrightarrow{PQ}$
3. (a) $m = -\frac{3}{4}, n = 7$ (b) $m = 4, n = -3$
4. $\overrightarrow{VW} = \frac{7}{2}\overrightarrow{XY}$
5. $k = 4$
6. $\overrightarrow{SR} = -\frac{8}{5}\overrightarrow{QT}$

Intensive Practice 8.1

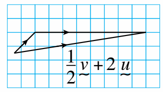
1. $\overrightarrow{AB} = 3\underset{\sim}{u}$
2. (a) 12 cm
   (b) (i) $\overrightarrow{EC} = 2\underset{\sim}{a}$ (ii) $\overrightarrow{BE} = 6\underset{\sim}{b}$
4. $h = -\frac{1}{2}, k = \frac{1}{2}$
5. $k = 4h - 2$

Self Practice 8.5

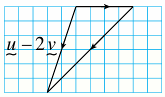
1. (a)



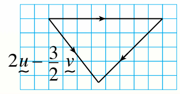
(b)



(c)



(d)



2. 131.19°, 106.30 km h^{-1}

3. (a) $\frac{2}{3}\underset{\sim}{y}$ (b) $-\underset{\sim}{x} + \underset{\sim}{y}$

(c) $\frac{1}{3}\underset{\sim}{y} - \underset{\sim}{x}$ (d) $-\frac{2}{3}\underset{\sim}{y} - \underset{\sim}{x}$

4. (a) 578.27 km h^{-1} (b) 345.07°

Self Practice 8.6

1. $k = \frac{30}{7}$

2. (a) $\overrightarrow{BD} = -24\underset{\sim}{x} + 20\underset{\sim}{y}$, $\overrightarrow{AE} = 6\underset{\sim}{x} + 15\underset{\sim}{y}$

Intensive Practice 8.2

1. (a) $\underset{\sim}{y} + \underset{\sim}{x}$ (b) $-\underset{\sim}{y} + \underset{\sim}{x}$

(c) $\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{y}$

2. (a) $3\underset{\sim}{x} + \underset{\sim}{y}$ (b) $\underset{\sim}{y} - 2\underset{\sim}{x}$

(c) $-\underset{\sim}{y} + 2\underset{\sim}{x}$

3. $-\underset{\sim}{a} + \underset{\sim}{b}$ **4.** $h = -10, k = 23$

5. 6.47 m s^{-1}

6. (a) (i) $-\underset{\sim}{b} + \underset{\sim}{a}$ (ii) $\frac{3}{5}(-\underset{\sim}{b} + \underset{\sim}{a})$

(iii) $\frac{2}{5}\underset{\sim}{b} + \frac{3}{5}\underset{\sim}{a}$ (iv) $-\underset{\sim}{b} + \frac{3}{4}\underset{\sim}{a}$

(b) (i) $\frac{2}{5}\lambda\underset{\sim}{b} + \frac{3}{5}\lambda\underset{\sim}{a}$ (ii) $(1 - \mu)\underset{\sim}{b} + \frac{3}{4}\mu\underset{\sim}{a}$

(c) $\lambda = \frac{5}{6}, \mu = \frac{2}{3}$

Self Practice 8.7

1. (a) $\overrightarrow{OA} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{OF} = -8\underset{\sim}{i}$, $\overrightarrow{BC} = -10\underset{\sim}{i} + \underset{\sim}{j}$,
$\overrightarrow{FA} = 10\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{DE} = 14\underset{\sim}{i}$, $\overrightarrow{DO} = -\underset{\sim}{j}$

(b) $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\overrightarrow{OF} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -10 \\ 1 \end{pmatrix}$, $\overrightarrow{FA} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$,
$\overrightarrow{DE} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$, $\overrightarrow{DO} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

2. (a) $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$ (b) 8.602 units

3. (a) (i) $\overrightarrow{AB} = 4\underset{\sim}{i} + \underset{\sim}{j}$ (ii) $\overrightarrow{BA} = -4\underset{\sim}{i} - \underset{\sim}{j}$

(iii) $\overrightarrow{BC} = -\underset{\sim}{i} - 5\underset{\sim}{j}$ (iv) $\overrightarrow{DC} = 2\underset{\sim}{i}$

(v) $\overrightarrow{AC} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ (vi) $\overrightarrow{AB} = 4\underset{\sim}{i} + \underset{\sim}{j}$

(b) $\overrightarrow{AB}$ and $\overrightarrow{DE}$, because they have the same gradient.

(c) $\overrightarrow{BA}$ and $\overrightarrow{DE}$, because they are in the opposite direction.

4. (a) $\underset{\sim}{p} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$
$\underset{\sim}{q} = -5\underset{\sim}{i} - 7\underset{\sim}{j}$
$\underset{\sim}{r} = \underset{\sim}{i} + 5\underset{\sim}{j}$

(b) $P(3, -4)$, $Q(-5, -7)$, $R(1, 5)$

(c) $|\underset{\sim}{p}| = 5$ units $|\underset{\sim}{q}| = 8.602$ units $|\underset{\sim}{r}| = 5.099$ units

Self Practice 8.8

1. (a) 3.606 units (b) 8.062 units

(c) $\frac{4}{7}$ units (d) 13 units

(e) 6 units

2. (a) $\frac{3\underset{\sim}{i} + 2\underset{\sim}{j}}{\sqrt{13}}$ (b) $\frac{-\underset{\sim}{i} - 9\underset{\sim}{j}}{\sqrt{82}}$

(c) $\underset{\sim}{i}$ (d) $\frac{-8\underset{\sim}{i} - 15\underset{\sim}{j}}{17}$

3. (a) Unit vector (b) Unit vector

(c) Unit vector (d) Not a unit vector

4. (a) ± 1 (b) ± 1

(c) 0 (d) $\pm\frac{1}{\sqrt{2}}$

(e) ± 0.866 (f) ± 0.988

5. $p = \pm 3$

6. $h = \pm\sqrt{2k - k^2}$

Self Practice 8.9

1. (a) $\begin{pmatrix} -9 \\ 30 \end{pmatrix}$ (b) $\begin{pmatrix} 16 \\ -47 \end{pmatrix}$

(c) $\begin{pmatrix} 12 \\ -13 \end{pmatrix}$ (d) $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$

2. (a) $10\underset{\sim}{i} + 18\underset{\sim}{j}$ (b) $2\underset{\sim}{i} + 6\underset{\sim}{j}$

(c) $-7\underset{\sim}{i} - 26\underset{\sim}{j}$ (d) $5.5\underset{\sim}{i} + 20\underset{\sim}{j}$

Self Practice 8.10

1. $\begin{pmatrix} 2 \\ -9.5 \end{pmatrix}$

2. Boat $A = \begin{pmatrix} 30 \\ 15 \end{pmatrix}$, Boat $B = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$
The two boat will not meet.

Intensive Practice 8.3

1. (a) $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ (b) 8.544 N

2. $k = 10$ or 1

3. $m = \frac{23}{3}$, $|\underset{\sim}{u}| : |\underset{\sim}{v}| = 9 : 16$

4. (a) $\overrightarrow{BC} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$ (b) $\frac{4\underset{\sim}{i} + 3\underset{\sim}{j}}{5}$

(c) $\overrightarrow{AR} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

5. 2.96 km h^{-1}, 101.69°

6. (a) 4, −8 (b) $\frac{6}{5}$

(c) 4

7. $\pm\frac{1}{\sqrt{2}}$ or $\pm\frac{\sqrt{2}}{2}$

8. $-2\sqrt{5}\,\underset{\sim}{i} + \sqrt{5}\,\underset{\sim}{j}$

9. $m = \frac{4 - n}{4}$

10. (a) $(50 - 4t)\underset{\sim}{i} + (20 + 4t)\underset{\sim}{j}$

(b) After 5 hours

MASTERY PRACTICE

1. (a) $\underset{\sim}{a} + \underset{\sim}{b}$ (b) $\underset{\sim}{a} - \underset{\sim}{c}$
2. $-\frac{2}{3}$
3. $m = \sqrt{1 - n^2}$
4. $h = \frac{2k + 17}{8}$
5. (a) $\frac{15\underset{\sim}{i} + 9\underset{\sim}{j}}{\sqrt{306}}$ (b) $C(18, 13)$
6. $\overrightarrow{RS} = \frac{2}{5}(3\underset{\sim}{i} - 2\underset{\sim}{j})$
7. $\overrightarrow{BC} = 2(\underset{\sim}{u} - \underset{\sim}{v})$
8. (a) (i) $-\underset{\sim}{a} + \underset{\sim}{b}$ (ii) $\underset{\sim}{b} - \underset{\sim}{a}$
   (iii) $2(\underset{\sim}{b} - \underset{\sim}{a})$ (iv) $\underset{\sim}{b} - 2\underset{\sim}{a}$
   (v) $2\underset{\sim}{b} - 3\underset{\sim}{a}$ (vi) $2(\underset{\sim}{b} - 2\underset{\sim}{a})$
   (b) $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{FC}$ (c) Parallel
9. (a) $\begin{pmatrix} 20 \\ -21 \end{pmatrix}$ (b) 29 km
   (c) $\begin{pmatrix} 30 \\ -32 \end{pmatrix}$
10. (a) (i) $6\underset{\sim}{u}$ (ii) $6\underset{\sim}{u} + 2\underset{\sim}{v}$
    (b) (i) $9\underset{\sim}{u} + 3\underset{\sim}{v}$ (ii) $9\underset{\sim}{u} + (2 + 3k)\underset{\sim}{v}$
    $k = \frac{1}{3}$
11. (a) (i) $4\underset{\sim}{a} + 4\underset{\sim}{c}$ (ii) $3\underset{\sim}{a} + 3\underset{\sim}{c}$
    (iii) $4\underset{\sim}{a} + 6\underset{\sim}{c}$ (iv) $\underset{\sim}{a} + 3\underset{\sim}{c}$
12. (a) The resultant velocity of Arul's boat is $4\underset{\sim}{i} + \frac{4}{3}\underset{\sim}{j}$
    The resultant velocity of Ben's boat is $7\underset{\sim}{i} + \frac{7}{3}\underset{\sim}{j}$
    The difference of the speed is 3.163 m s^{-1}
    (b) $\frac{3\underset{\sim}{i} - \underset{\sim}{j}}{\sqrt{10}}$

CHAPTER 9 SOLUTION OF TRIANGLES

Self Practice 9.1

1. (a) $\frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$
   (b) $\frac{k}{\sin K} = \frac{l}{\sin L} = \frac{m}{\sin M}$
   (c) $\frac{6}{\sin 40°} = \frac{8}{\sin 120°}$

Self Practice 9.2

1. (a) 5.611 cm (b) 52.29°
   (c) 9.331 cm
2. 55.344 m

Self Practice 9.3

1. (a) Ambiguous case exists.
   (b) Ambiguous case does not exist.
2. (a) 57.86° or 122.14°
   (b) 7.112 cm or 18.283 cm

Self Practice 9.4

1. 10.147 m
2. 41.224 m

Intensive Practice 9.1

1. $\angle A = 64°$, $a = 37.359$ cm, $c = 26.158$ cm
2. (a) $BE = 8$ cm, $CE = 6$ cm, $DE = 15$ cm
   (b) $\angle EAB = 53.13°$, $\angle BCE = 53.13°$, $\angle BCD = 126.87°$, $\angle ABD = 81.20°$, $\angle CBD = 25.06°$
   (c) Triangle *BDC* and triangle *BDA* have same angle and two sides with same length.
3. (a) $\angle PQR = 120°$ (b) 5.529 cm
4. 61.62 cm
5. 138.58 m

Self Practice 9.5

1. (a) 3.576 cm (b) 18.661 cm
   (c) 53.891 m
2. (a) 51.38° (b) 35.26°
   (c) 99.06°
3. 69.93°

Self Practice 9.6

1. 29.614 m
2. 41.832 m
3. 48.046 km

Intensive Practice 9.2

1. 4.071 cm, 6.475 cm
2. 11.555 km
3. 46.50°
4. 23.974 m

Self Practice 9.7

1. (a) 112.482 cm^2 (b) 28.670 cm^2
   (c) 75.206 cm^2
2. 27.078 cm
3. 51.237 cm^2
4. 18.146 m^2

Self Practice 9.8

1. 16.142 cm^2
2. 17.69 cm^2
3. 2

Self Practice 9.9

1. 251.716 m^2
2. 66.169 cm^2

Intensive Practice 9.3

1. (a) 6 cm (b) 6 cm^2
2. 43.012 cm^2
3. 7.501 cm or 17.713 cm
4. 107.977 cm^2
5. 89.233 cm^2
6. 14.664 cm

Self Practice 9.10

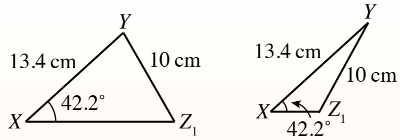
2. (a) 19.519 cm (b) 115.865 cm^2
3. 98.13°, 3.5 $units^2$

Intensive Practice 9.4

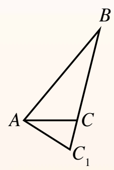
1. (a) 40.20 cm^2 (b) 125.63°
2. 9.266 km
3. (a) 31.241 cm^2 (b) Plane *DBR*
4. 31.455 km, 187.11°
5. 457.803 m

MASTERY PRACTICE

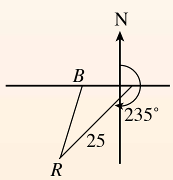
1. (a) $a = 6.504$ cm, $b = 5.239$ cm
   (b) $\angle P = 105.03°$, $\angle Q = 49.92°$, $\angle R = 25.05°$
2. (a) 6.756 cm (b) 7.287 cm
3. (a) 13.82 cm (b) 33.387 cm^2
4. (a)



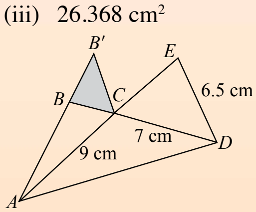
   (b) 64.17°, 115.83° (c) 25.066 cm^2
5. (a) 5.903 cm (b) 42.66°
6. (a) 37.59° (b) 14.309 cm^2
7. (a) 118.9° (b)



   (c) 5.142 m
8. (a) 40° (b) 3.875 cm
   (c) 5.763 cm^2
9. (a)



   (b) 13.38 km
   (c) (i) 14.20 km (ii) 153.36°
10. (a) (i) 58.277 km (ii) 58.76°
   (iii) 2535.79 km^2
   (b) Petrol station M (c) 63.395 km
11. (a) (i) 124.35° (ii) 6.943 cm
   (iii) 26.368 cm^2
   (b)



12. (a) $\frac{10}{33}$ (b) 17.762 cm^2, 8.881 cm
   (c) Triangle ZXY' such that XZ is the same, $XY' = XY$, $\angle XZY' = \angle XZY$
   Triangle ZXY' such that XZ is the same, $ZY' = ZY$, $\angle ZXY' = \angle ZXY$

CHAPTER 10 INDEX NUMBERS

Self Practice 10.1

1. $I = 82.20$
   The number of registered commercial vehicles decreased by 17.80% in the year 2017 compared to the year 2015.
2. $I = 112.72$
   The average monthly expenditure of a household increased by 12.72% in the year 2017 compared to the year 2014.
3. 650 053 107 metric tonnes
4. 150
5. 94.48

Self Practice 10.2

1. 112 2. 104.76

Intensive Practice 10.1

1. $I = 108.3$
   The average temperature in town P increased by 8.3% in February 2017 compared to January 2017.
2. $I = 92.31$
   The price of a certain item decreased by 7.69% in the year 2015 compared to the year 2012.
3. $x = 0.5$, $y = 2.80$, $z = 125$
4. $p = 100$, $q = 131.90$, $r = 134.48$
   $s = 125.86$
5. 107.27

Self Practice 10.3

1. 105 2. 114

Self Practice 10.4

1. (a) $I_A = 150$, $I_B = 104$, $I_C = 120$, $I_D = 124$
   (b) 121
   There was a 21% increase in the price of all goods in the year 2016 compared to the year 2010.
   (c) RM2.19
2. (a) $a = 115$, $b = 150$, $c = 112.5$, $d = 33$
   (b) 126.68 (c) RM44.34
   (d) 110

Intensive Practice 10.2

1. (a) 124 2. 93
3. 76.4
4. (a) 130 (b) 132
   (c) RM25.74

MASTERY PRACTICE

1. (a) $x = 1.00$, $y = 1.00$, $z = 110$
   (b) 112.5
2. $m = 121$, $n = 122$
3. (a) RM9.12 (b) 35 000
   (c) 90.4%
4. (a) 64.12 (b) 0.935 million tonnes
   (c) 87.15
5. (a) RM 15 (b) 187.5
6. (a) 4 (b) 105.25
7. (a) 133.03
8. $p = 140$, $q = 130$, $r = 255$
9. (a) 6.14 million
   (b) 166.85
   The number of visitors in the year 2020 increases by 66.85% compared to the year 2017.
10. $x : y : z = 1 : 4 : 3$
11. (a) $P_{2014} = \text{RM}150.91$, $P_{2010} = \text{RM}188.64$
   (b) 12%
12. (a) 115 (b) RM198.38

GLOSSARY

Ambiguous case (*Kes berambiguiti*); Having more than one solution in a triangle.

Base (*Asas*); If a is a number and is written in index form, for example a^n, then a is the base.

Base year (*Tahun asas*); A year that is chosen as a starting point for the calculation of a series of index numbers, usually a year that has normal characteristics.

Codomain (*Kodomain*); A set, of which part of it is mapped from a domain set.

Common difference (*Beza sepunya*); A constant that is added to the previous term to form an arithmetic progression.

Common ratio (*Nisbah sepunya*); A constant that is multiplied to the previous term to form a geometric sequence.

Composite function (*Fungsi Gubahan*); Function of another function.

Conjecture (*Konjektur*); A prediction that is not proven but seems true. If there is sufficient proof, the prediction becomes a theorem or a formula.

Continuous function (*Fungsi selanjar*); A function with points on a graph that are connected with a line or curve in a certain interval.

Discrete function (*Fungsi Diskret*); A function with points on a graph, which points are not connected with a line or curve.

Domain (*Domain*); Set of elements that are mapped to another set by a relation.

Function (*Fungsi*); A special relation in which every object in the domain corresponds to exactly one element of the range.

Heron formula (*Rumus Heron*); A formula that is used to determine the area of a triangle when the length of all sides are known.

Horizontal line test (*Ujian garis mengufuk*); A horizontal line that is used to identify if a function is a one-to-one function.

Included angle (*Sudut kandung*); Angle between two given sides of a shape.

Index (*Indeks*) If a is a number, n is a positive integer and a^n, then n is the index.

Index number (*Nombor indeks*); A number that expresses the relative change of a quantity with respect to time.

Inverse function (*Fungsi songsang*); A function that maps every image in the function to its object.

Line of best fit (*Garis lurus penyuaian terbaik*); The best straight line that is drawn from points that do not form a perfect straight line.

Line segment (*Tembereng garis*); A part of a line that connects two end points.

Linear equation (*Persamaan linear*); An equation that satisfies $y = mx + c$ and forms a straight line.

Linear equation in three variables (*Persamaan linear dalam tiga pemboleh ubah*); An equation in the form of $ax + by + cz = d$, where a, b and c are constants and non-zero.

Locus (*Lokus*); A point that moves with a path that is traced by those points according to certain conditions.

Logarithms (*Logaritma*); The logarithm of a positive number N to a positive base a is the index for a, i.e., if $N = ax$, then $\log_a N = x$.

Nonlinear equation (*Persamaan tak linear*); An equation whose highest power of a variable is more than one.

One to one function (*Fungsi satu dengan satu*); A relation whereby each object has only one image.

Parallel line (*Garis selari*); Two or more lines with the same gradient.

Parallel vector (*Vektor selari*); Two vectors are parallel if one vector is a scalar multiple of another vector.

Perpendicular line (*Garis serenjang*); Two lines that intersect at 90 degrees.

Plane (*Satah*); A flat surface that consists of a horizontal plane, vertical plane and curved plane.

Power (*Kuasa*); If a is a number and n is a positive integer, then a^n is a number and is called a power of n.

Price index (*Indeks harga*); A statistical measurement that is used to show price changes in a certain time.

Progression (*Janjang*); A sequence of numbers that is formed by adding or multiplying a constant to the previous term (except the first term).

Quadratic function (*Fungsi kuadratik*); A function in the form of $f(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic inequality (*Ketaksamaan kuadratik*); An inequality with one quadratic expression with a variable on the left and zero on the other side.

Quadratic equation (*Persamaan kuadratik*); An equation in the form of $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

Resultant vector (*Vektor paduan*); A single vector that is formed from a combination of two or more vectors.

Range (*Julat*); Subset of the codomain that contains all the images which have been mapped by the objects in the domain.

Root (*Punca*); The value of an unknown which satisfies an equation.

Simultanenous equation (*Persamaan serentak*); Two or more equations that contain common variables.

Tangent (*Tangen*); A straight line that touches a curve at a point without crossing over.

Term (*Sebutan*); Numbers that form a numerical sequence or progression.

Terminal point (*Titik terminal*); The endpoint on a line segment that represents a vector.

Unit vector (*Vektor unit*); A vector with magnitude of one unit at a certain direction.

Variable (*Pemboleh ubah*); A quantity whose value is unknown and not fixed.

Vector magnitude (*Magnitud vektor*); The length or size of a vector.

Vector quantity (*Kuantiti vektor*); A quantity that has magnitude and direction.

Vertex (*Verteks*); The minimum or maximum point of a parabola.

Vertical line test (*Ujian garis mencancang*); A vertical line that is used to determine whether the relation of a graph is a function.

Weightage (*Pemberat*); A constant assigned to a single item, indicative of the item's relative importance.

Zero vector (*Vektor sifar*); A vector that has zero magnitude and undefined direction.

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Dengan ini **SAYA BERJANJI** akan menjaga buku ini dengan baiknya dan bertanggungjawab atas kehilangannya, serta mengembalikannya kepada pihak sekolah pada tarikh yang ditetapkan

Skim Pinjaman Buku Teks

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